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PHD DISSERTATION

Mathematical analysis of a new model of bone pattern formation

Paramita Chatterjee

Supervisor
Bogdan Kaźmierczak, Institute Professor
Institute of Fundamental Technological Research
Polish Academy of Sciences

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Abstract

In this work we study a mathematical model proposed in reference [16] to describe processes of pattern formation in morphogenesis. In fact, the model is dedicated to bone formation phenomena in vertebrate embryos. The first equation of the considered system does not fall into any of the three basic classes of partial differential equations. To be more precise, this equation is parabolic in time and space and hyperbolic with respect to time and a pair of auxiliary independent variables describing the state of the cells. This fact does not allow us to use, at least straightforwardly, the usual methods of analysis assigned either to strictly parabolic or strictly hyperbolic problems and, according to our knowledge based on literature search and private communications, there are no theorems guaranteeing the existence of such equations even the homogeneous case. Similar difficulties occur when we attempt to carry out the numerical simulations of the model.

Our study of the problems connected with the analysed system is divided into two parts. In first part, see Part II, we consider a scalar equation retaining the basic difficulties of the system. We do not take into account the non-local terms (which are the source of aggregation phenomena), but concentrate on the existence of solutions to linear equations, homogeneous as well as inhomogenous ones. We manage to construct the solutions by means of appropriately defined solution kernels, both in the spatially unbounded as well as bounded case. We prove that the constructed solutions are unique in appropriate spaces of functions. We can also show the validity of the expressions defining solutions to homogeneous equations, when the initial data are given in the product form and the problem can be solved starightforwardly. In second part, see Part III, we deal with the whole system of three equations describing the analysed system, however we use another approach to prove the existence of its solutions. The approach consists in assigning to this system a modified version of the Rothe numerical scheme with time interval discretized into intervals of the lenght Δt. By deriving a series of a priori estimates, we are able to prove that the proposed numerical scheme produces, in the limit $\Delta t \to 0$, a solution to the system, in which, similarly to Part II we replace the non-local term by local functions.

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Part I
Introduction

1 Background and motivation

Morphogenesis is one of the most interesting phenomenon in biology. It is extremely intriguing to explain, how from an initially homogeneous set of identical cells, spatial patterns composed of differentiated cells, leading to the formation of tissues organs and finally whole organisms can be created.

One of the important example of morphogenetic processes is the vertebrate limb formation. The formation of the skeletal pattern in vertebrate limbs received particular attention by the researchers. To be more precise, the mechanism of cellular and molecular interactions during the growth of the avian forelimb, for example, spatio-temporal differentiation of cartilage, such that the number of bone primordial changes in time from one (humerus), to two (radius and ulna) and to three (digits) get significant attention. Here we should keep in mind that the mechanism of chondrogenesis may differ from species to species, but its main features are common to all the vertebrates. In the experimental context the bone formation process is most often described for mice or chickens. At the initial stages of embryo development, limb mesenchymal cells started to condense to so called precartilage. After then the precartilage mesenchymal cells organize themselves into spot- or rod-like condensations of nearly uniform size [8, 7, 22], which are then turn into definite cartilage, followed by bone. These phenomena, together with appropriate geometry of the limb bud have been the subjects of many biological as well as mathematical models (see e.g. the reviews [34] , [27], [28]). In Appendix B (after the **References**), we attach a review by P. Chatterjee, T. Glimm & B. Kazmierczak [5], submitted to the journal of Mathematical Biosciences, which, among others, relates the considered model to other models of pattern formation, especially to the one presented in [2]. (However, let us emphasize that [5] is not a part of this dissertation and has been appended only for completeness.) The nature of the bone pattern formation, especially in its initial stages is still being not fully recognized and there are different candidates for proteins responsible for the onset of this process. In an experimental paper [3], Bhat et al. suggested that two members of a class of glycan-binding proteins CG(chicken galectin)-1A and CG-8 play a crucial role in cell condensation in the developing chicken limb. During the experiment, it was observed by Bhat et al. that, in vitro, CG-1A promotes supernumerary condensation formation and in vivo, it induces digit formation, while CG-8 inhibits both of these processes. Also, CG-1A induces the expression of the receptor, which binds both of CG-1A and CG-8 (the shared receptor).

In [16], Glimm et al., proposed a mathematical model describing the interactions of CG-1A and CG-8, based on the above mentioned experiment. It was verified in [16], that the proposed model reproduces well the experimental findings.

Mathematical formulation of the considered model

The model describes the spatio-temporal evolution of the following quantities:

1. $c_1^u = c_1^u(t, \mathbf{x})$ - concentration of freely diffusible CG-1A (that is, CG-1A not bound to receptors on cell membranes),
2. $c_8^u = c_8^u(t, \mathbf{x})$ - concentration of freely diffusible CG-8 (that is, CG-8 not bound to receptors on cell membranes)
3. $R = R(t, \mathbf{x}, c_1, c_8^8, c_8^1, \ell_1, \ell_8)$ - cell density.

Let us note that the cell density R depends on several variables representing various chemical concentrations **besides** to time and space, that is to say: c_1 - concentration of CG-1A proteins bound to shared receptors on cell membranes, c_8^8 - concentration of CG-8 proteins bound to CG-8 receptors on cell membranes, c_8^1 - concentration of CG-8 proteins bound to shared receptors on cell membranes, ℓ_1 - concentration of shared receptors (not bound to galectins) on cell membranes and ℓ_8 - concentration of CG-8 receptors (not bound to galectins) on cell membranes.

In [16] the following system of equations is proposed for $t \in (0, T)$, x from some bounded domain $\Omega \subset \mathbb{R}^{n_\Omega}$, $n_\Omega \geq 1$, $(c_1, c_8^8, c_8^1, \ell_1, \ell_8) \in (0, \infty)^5$:

$$\frac{\partial R}{\partial t} = \underbrace{D_R \nabla^2 R}_{\text{cell diffusion}} - \underbrace{\nabla \cdot (R\,\mathbf{K}(R))}_{\text{cell-cell adhesion}}$$

$$\underbrace{-\frac{\partial}{\partial c_1}(\alpha R) - \frac{\partial}{\partial c_8^8}(\beta_8 R) - \frac{\partial}{\partial c_8^1}(\beta_1 R)}_{\text{binding/unbinding of galectins to receptors}} - \underbrace{\frac{\partial}{\partial \ell_1}\left[(\lambda - \alpha - \beta_1)R\right] - \frac{\partial}{\partial \ell_8}\left[(\delta - \beta_8)R\right]}_{\text{change in receptors}} \tag{1.1}$$

$$\frac{\partial c_1^u}{\partial t} = \underbrace{D_1 \nabla^2 c_1^u}_{\text{diffusion}} \underbrace{+\overline{\nu} \int c_8^8 R\, dP}_{\substack{\text{pos. feedback of CG-8} \\ \text{on prod. of CG-1A}}} \underbrace{- \int \alpha R\, dP}_{\substack{\text{binding of CG-1A} \\ \text{to its receptor}}} \underbrace{-\overline{\pi}_1 c_1^u}_{\text{degradation}} \tag{1.2}$$

$$\frac{\partial c_8^u}{\partial t} = \underbrace{D_8 \nabla^2 c_8^u}_{\text{diffusion}} \underbrace{+\overline{\mu}\, c_1 R\, dP}_{\substack{\text{pos. feedback of CG-1A} \\ \text{on prod. of CG-8}}} \underbrace{- \int \beta_1 R\, dP - \int \beta_8 R\, dP}_{\substack{\text{binding of CG-8} \\ \text{to receptors}}} \underbrace{-\overline{\pi}_8 c_8^u}_{\text{degradation}} \tag{1.3}$$

subject to the following initial and boundary conditions:

$$R(0, \mathbf{x}, c_1, c_8^8, c_8^1, \ell_1, \ell_8) = R_0(\mathbf{x}, c_1, c_8^8, c_8^1, \ell_1, \ell_8) \tag{1.4}$$

$$\frac{\partial R}{\partial \mathbf{n}} = 0 \text{ for } \mathbf{x} \in \partial\Omega,\ R\big|_{c_1=0} = R\big|_{c_8^8=0} = R\big|_{c_8^1=0} = R\big|_{\ell_1=0} = R\big|_{\ell_8=0} = 0 \tag{1.5}$$

$$\frac{\partial c_1^u}{\partial \mathbf{n}} = \frac{\partial c_8^u}{\partial \mathbf{n}} = 0 \text{ for } \mathbf{x} \in \partial\Omega, \tag{1.6}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ denotes the unit outward vector to the boundary $\partial\Omega$ and

$$\frac{\partial}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla_x \ .$$

In (1.2) and (1.3)

$$\int dP = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty dc_1 dc_8^8 dc_8^1 d\ell_1 d\ell_8.$$

Remark Let us emphasize that in Eqs. (1.1)-(1.6), the quantities c_1, c_8^8, c_8^1, ℓ_1, ℓ_8 are treated as independent variables **similarly to time t and space $\mathbf{x}$** . □

The cell-cell adhesion term is assumed to have the form:

$$\mathbf{K} = \Psi\Big(\nu; dist(\mathbf{x}, \partial\Omega)\Big)\overline{\alpha}_K\, c_1 \int\int_{D_{\rho_0}(\mathbf{x})} \widetilde{c}_1\, \sigma(R(t, \mathbf{x}+\mathbf{r}, \tilde{c}_1, \tilde{c}_8^8, \tilde{c}_8^1, \tilde{\ell}_1, \tilde{\ell}_8))\, dP\, \frac{\mathbf{r}}{|\mathbf{r}|}\, d\mathbf{r} \tag{1.7}$$

Here $\overline{\alpha}_K$ is a constant which represents the strength of the adhesion, whereas for some $\nu > 0$ sufficiently small, $\Psi(\nu; \cdot)$ is a smooth, monotone cut-off function such that $\Psi(\nu; y) \equiv 1$ for $y \geq 2\nu$ and $\Psi(\nu; y) \equiv 0$ for $y \leq \nu$. For example, we can take

$$\Psi(\nu; y) := \begin{cases} 0 & y \in (0, \nu] \\ \dfrac{\Psi_*(y-\nu)}{\Psi_*(y-\nu) + \Psi_*(2\nu - y)} & y \in (\nu, 2\nu) \\ 1 & y \geq 2\nu\,, \end{cases} \tag{1.8}$$

where

$$\Psi_*(s) = \begin{cases} e^{-\frac{1}{s}}, & s > 0 \\ \\ 0, & s \leq 0\,. \end{cases}$$

The function $\sigma(R)$ describes the dependence of the adhesion forces on the cell density.

We concentrate here only on showing the structure of the proposed equations. The precise expressions of the terms entering the above system can be found in [16].

Model modifications

As we noted above, the independent variables of system (1.1)-(1.2)-(1.3) are $t, \mathbf{x}, c_1, c_8^8, c_8^1, \ell_1$ and ℓ_8. In [16], using time scale separation, a simpler set of equations based on the assumption of fast receptor binding and unbinding has been proposed.

Let T_1 denote the total concentration of CG-1A receptors (whether unbound or bound to CG-1A or CG-8), i.e.

$$T_1 = c_1 + c_8^1 + \ell_1. \tag{1.9}$$

Similarly, the total concentration of CG-8 receptor can be defined as:

$$T_8 = c_8^8 + \ell_8. \tag{1.10}$$

Under the assumption that the process of 'galectin binding' is very fast, we obtain (after non-dimensionalization procedure) the following system of equations presented in [16]:

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \nabla \cdot (R\,\mathbf{K}(R)) - \frac{\partial}{\partial T_1}\left(\widetilde{\gamma}(c_1^u, c_8^u, T_1)\, R\right) - \frac{\partial}{\partial T_8}\left(\widetilde{\delta}(c_8^u, T_8)\, R\right) \tag{1.11}$$

$$\frac{\partial c_1^u}{\partial t} = \nabla^2 c_1^u + \widetilde{\nu} \int_0^\infty \int_0^\infty c_8^8\, R\, dT_1\, dT_8 - c_1^u \tag{1.12}$$

$$\frac{\partial c_8^u}{\partial t} = \nabla^2 c_8^u + \widetilde{\mu} \int_0^\infty \int_0^\infty c_1\, R\, dT_1\, dT_8 - \widetilde{\pi}_8\, c_8^u \tag{1.13}$$

with

$$c_8^8 = c_8^8(t, \mathbf{x}, T_8) = \frac{c_8^u T_8}{1 + c_8^u}, \quad c_1 = c_1(t, \mathbf{x}, T_1) = \frac{c_1^u T_1}{1 + f c_8^u + c_1^u} \tag{1.14}$$

$$\widetilde{\gamma}(c_1^u, c_8^u, T_1) = \left(\frac{2 c_1^u}{\frac{c_1^u T_1}{c_1^u + f c_8^u + 1} + \widetilde{c}_1} - \widetilde{\gamma}_2 \right) \frac{T_1}{c_1^u + f c_8^u + 1}, \quad \widetilde{\delta}(c_8^u, T_8) = 1 - \widetilde{\delta}_2 \frac{T_8}{1 + c_8^u} \tag{1.15}$$

$$\mathbf{K}(t, \mathbf{x}, T_1, R(t; \cdot)) =$$

$$\Psi\Big(\delta; dist(\mathbf{x}, \partial\Omega)\Big) \widetilde{\alpha}_K\, c_1(t, \mathbf{x}, T_1) \int_0^\infty \int_0^\infty \int_{D_{r_0}(\mathbf{x})} c_1(t, \mathbf{s}, \tilde{T}_1) \widetilde{\sigma}(R(t, \mathbf{s}, \tilde{T}_1, \tilde{T}_8)) \frac{\mathbf{s}}{|\mathbf{s}|}\, ds\, d\tilde{T}_1\, d\tilde{T}_8 \tag{1.16}$$

Here one can either use a linear $\widetilde{\sigma}(R) = R$ or logistic form for $\widetilde{\sigma}$ in the expression for the adhesion flux

$$\widetilde{\sigma}(R) = \eta_\sigma \max\left(1 - \frac{1}{\widetilde{R}_m} \int_0^\infty \int_0^\infty R\, dT_1\, dT_8, 0\right), \tag{1.17}$$

where η_σ and $\widetilde{R}_m$ are positive constants. According to (1.5), the following boundary conditions hold:

$$\frac{\partial R}{\partial \mathbf{n}} = 0 \text{ for } \mathbf{x} \in \partial\Omega,\ R\big|_{T_1=0} = R\big|_{T_8=0} = 0. \tag{1.18}$$

System (1.11)-(1.13) was analysed numerically in [16]. It was shown that the system is possible to generate periodic structures.

In system (1.11)-(1.13), the quantity R denotes the concentration of cells at time t and at a given point in $\Omega \times \mathbb{R}^2_{T_1 T_8}$. Hence the spatial concentration of cells at a given point $\mathbf{x} \in \Omega$ should be calculated as an integral over the whole $\mathbb{R}^2_{T_1T_8}$, in fact over its non-negative quadrant P_{18} of this space. That is to say:

$$R^x(t, \mathbf{x}) = \int_{P_{18}} R(t, \mathbf{x}, T_1, T_8) dT_1 dT_8. \tag{1.19}$$

2 Specificity of the system and the objective of the dissertation

As T_1 and T_8 are independent variables, then even for given functions c_1^u and c_8^u, Eq.(1.11) is not parabolic. Due to the form in which T_1 and T_8 enter this equation, we can say that it is of mixed parabolic-hyperbolic type. According to *our* best knowledge, as well as to the opinions of the leading specialists in partial differential equations expressed in private communications, there is no general theory of such equations. Thus we cannot 'a priori' guarantee the existence and uniqueness of solutions even locally in time. The situation is complicated by the presence of the non-local adhesion term. It should be noticed that the simplified system given by Eqs (1.11)-(1.13) inherit qualitatively the same difficulties as the full system (1.1)-(1.3).

The presence of the hyperbolic terms in Eq.(1.11) can be formally justified by means of the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho v) = 0$$

applied to the flow in the space (T_1, T_8). Let us start from the continuity equation in the space T_1. Identifying density ρ with R and T_1 with the spatial variable x_1, and v with $\dfrac{\partial T_1}{\partial t}$, we can write the continuity equation in the form:

$$\frac{\partial R}{\partial t} + \frac{\partial\left(R \frac{\partial T_1}{\partial t}\right)}{\partial T_1} = 0.$$

As the speed of the flow in the space T_1, $\dfrac{\partial T_1}{\partial t}$ can be interpreted as the rate of production of T_1. Likewise, in the space (T_1, T_8) we can write the continuity equation in the form:

$$\frac{\partial R}{\partial t} + \nabla_{T_1,T_8} \cdot \left(R\left(\frac{\partial T_1}{\partial t}, \frac{\partial T_8}{\partial t}\right)\right) = 0,$$

where the components of the vector $\left(\dfrac{\partial T_1}{\partial t}, \dfrac{\partial T_8}{\partial t}\right)$ can be interpreted as the rates of production of T_1 and T_8 quantities. The validity of these arguments can be seen by noticing the correspondence of the terms $\dfrac{\partial}{\partial T_1}\left(\widetilde{\gamma}(c_1^u, c_8^u, T_1)\, R\right)$ and $\dfrac{\partial}{\partial T_8}\left(\widetilde{\delta}(c_8^u, T_8)\, R\right)$ in the Eq.(1.11) with the terms in the second line of Eq.(1.1). In Eq.(1.11), these rates are denoted as $\widetilde{\gamma}$ and $\widetilde{\delta}$ and depend additionally on the quantities c_1^u and c_8^u. In a sense, the presence of the hyperbolic like terms is similar as in equations describing population dynamics models.

The objective of this dissertation is to prove at least local in time existence theorems for system (1.11)-(1.13). To be more precise, in our study we will concentrate on further simplified forms of system (1.11)-(1.13). The main simplification consists in the fact that the integral term will be either ignored (as in Part II) or replaced by a local terms depending on R and its gradient ∇R. This approach can be justified by the fact that the existence problem of Eq.(1.11) without the hyperbolic like terms were considered in the papers [9], [10]. Moreover, having the local existence theorems for the equation without the non-local integral terms $\nabla \cdot (R\,\mathbf{K}(R))$, we can study the existence of system (1.11)-(1.13) in its full generality.

The analysis of system (1.11)-(1.13) is divided into two parts, which are distinguished according to the approaches used. In Part II we confine ourselves to a scalar equation, representing the considered system with the non-local integral term replaced by a given function of (t, x, T_1, T_8). Beside to this, to obtain an initial insight, in sections 3-9, we introduce additional simplification consisting in modelling the investigated biological object representing the limb bud by the whole space. (Of note, this kind of approach has been used in the papers [9], [10].) For technical reasons, we also restrict ourselves to the functions $\widetilde{\gamma}$ and $\widetilde{\delta}$ depending *mainly* only on T_1 and T_8 respectively and independent of t. (In sections 4 and 13, we allow the function Γ and B to depend also on t.) The advantage of introducing these simplifying conditions are mainly manifested in the fact that we are able to obtain solutions in explicit form, which can be relatively easy to analyse. In section 11, using section 10, we formulate the existence results in bounded regions. In section 4, we discuss the uniqueness of solutions Eq.(3.1) and its inhomogeneous counterpart. In section 13, we establish natural generalizations of the obtained results to the case of any number of T-variables.

The strong simplifying conditions imposed on equations of system (1.11)-(1.13) are essentially relaxed in the Rothe method approach applied in Part III. In fact, in Part III we analyse system (16.2)-(16.4), which differs from (1.11)-(1.13) only by replacement of the non-local (integral) term by a given function of R and the components of ∇R. For such a system of differential equations, it is possible to prove the existence of solutions to system (16.2)-(16.4) by showing the convergence of the solutions to the systems with discretized time as the time step $\Delta t \to 0$. The precise description of the methods applied in Part III is given in section 16.2 and we will not repeat it here.

Part II

Linearized scalar equation representing system (1.11)–(1.13). The Green's function approach

3 The case of $\tilde{\gamma}$ and $\tilde{\delta}$ independent of x and t

As we mentioned above, due to the presence in Eq.(1.11) of the convective terms with respect to T_1 and T_8, we cannot *a priori* guarantee the existence and uniqueness of solutions to system (1.11)-(1.13) even for small times. To get some preliminary insight, we will consider in this section linear scalar equations, which can be regarded as linearised forms of Eq.(1.11). We start from the simplest equation retaining the parabolic-hyperbolic features of Eq.(1.11), i.e. for Γ and B being linear functions of T_1 and T_8 respectively, but then consider more general cases. In subsection 5 we consider weak asymptotics of solutions to homogeneous equations of the form (3.1) with respect to a scaling parameter λ describing the magnitude of the functions Γ and B (see subsection 5.1, in particular Lemma 5.5), as well as similar weak asymptotics of solutions to non-homogeneous equations of the form (3.55) (see subsection 5.2). Lemma 5.5 can be considered as a partial justification of the reduced system proposed formally in [16] as a radical approximation of system (1.11)-(1.13). These results are rewritten in section 6, where weak formulation of Eq.(3.1) has been considered.

It seems that the **main result** of Part II is the construction of the solution to Eq.(3.1) and its inhomogeneous version (3.55). This solution corresponds to a convolution of the two semigroups. As we mentioned above, to establish this fact, we examined a couple of cases, starting from the simplest possible case and then increasing the generality of the functions Γ and B, but we allowed ourselves to retain these cases in the dissertation. The **second important result** concerns the possibility of arriving at the solution to Eq.(3.55) from a parabolic equation obtained by adding small diffusional terms with respect to T_1 and T_8. To be more precise, in section 9 we state that adding diffusional terms $\varepsilon^2(R_{,T_1T_1} + R_{,T_8T_8})$ does not change the properties of the solutions, which tend in the space of smooth functions to the solutions for $\varepsilon = 0$.

If $\tilde{\gamma}$ and $\tilde{\delta}$ do not depend on c_1^u and c_8^u, i.e. when $\tilde{\gamma}(c_1^u, c_8^u, T_1) = \Gamma(T_1)$, $\tilde{\delta}(c_8^u, T_8) = B(T_8)$ and $K(R) = 0$, then the first equation of system (1.11)-(1.13) becomes separated. Let us suppose additionally, for convenience, that $\Omega \equiv \mathbb{R}^3$. In this case, Eq.(1.11) takes the form

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1}(\Gamma(T_1)\, R) - \frac{\partial}{\partial T_8}(B(T_8)\, R) \qquad (t, \mathbf{x}, (T_1, T_8)) \in (0, T] \times \mathbb{R}^3 \times \mathbb{R}_+^2, \tag{3.1}$$

together with the boundary conditions

$$R(t, \mathbf{x}, T_1, T_8) = 0 \qquad \text{for } \{(t, x, T_1, T_8) : T_1 = 0 \vee T_8 = 0\}.$$

Above and below we use the following denotations of the positive (non-negative) subsets of the real axis and the plane:

$$\mathbb{R}_+ := \{r \in \mathbb{R} : r > 0\}, \qquad \overline{\mathbb{R}_+} := \{r \in \mathbb{R} : r \geq 0\},$$

and

$$\mathbb{R}_+^2 := \{(r_1, r_8) \in \mathbb{R}^2 : r_1 > 0, r_8 > 0\}, \qquad \overline{\mathbb{R}_+^2} := \{(r_1, r_8) \in \mathbb{R}^2 : r_1 \geq 0, r_8 \geq 0\}. \tag{3.2}$$

Before proceeding, let us formulate an obvious conservation law.

Lemma 3.1. *Let the initial data $R_0(x, T_1, T_8)$ satisfy the equality*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^2_+} R_0(x, T_1, T_8) dT_1 dT_8 \, dx = M_0.$$

Suppose that a solution $R : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^2_+$ to Eq.(3.1) satisfies the conditions $R(t, x, T_1, T_8) \equiv 0$ for $T_1 = 0$ or $T_8 = 0$ and that $\nabla_x R = o(\|\mathbf{x}\|^{-2})$ as $\|\mathbf{x}\| \to \infty$ and $R = o((|T_1|+|T_8|)^{-1})$ as $|T_1|+|T_8| \to \infty$. Then for all $t \in [0, T]$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^2_+} R(t, x, T_1, T_8) dT_1 dT_8 \, dx = M_0.$$

Proof Let us note that the proof follows by considering the improper integrals over the set $\mathbb{R}^3 \times \mathbb{R}^2 \ni (x, T_1, T_8)$ of the both sides of Eq.(3.1) using Fubini's and Gauss-Ostrogradskii theorems. In fact, the integration of the left hand side gives

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2_+} R(t, x, T_1, T_8) dT_1 dT_8 \, dx,$$

whereas the values of the integrals of the right hand sides over the sets $B^3(0, r) \times \left(B^2(0, r) \cap R^2_+\right)$, where $B^k(0, r)$ denotes a k-dimensional open ball with centre at 0 and the radius r, tend to 0 as $r \to \infty$. $\square$

Similarly to Eq.(1.11), Eq.(3.1) is of mixed type. It is parabolic in the direction of spatial variables $\mathbf{x} = (x_1, x_2, x_3)$ and hyperbolic in the direction (T_1, T_8). To begin with, let us analyse the possible characteristic curves of Eq.(3.1). Our discussion will be based upon [32, chapter 2]. The principal part of the operator

$$P(t, x_1, x_2, x_3, T_1, T_8; D) := -\frac{\partial R}{\partial t} + d_R \nabla^2 R - \frac{\partial}{\partial T_1} \left(\Gamma(T_1) \, R\right) - \frac{\partial}{\partial T_8} \left(B(T_8) \, R\right)$$

(acting on R) is equal to

$$d_R \, \nabla^2 = d_R \left(\left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2 \right).$$

Let, at a given point $(t, x_1, x_2, x_3, T_1, T_8) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^2$,

$$\sigma(t, x_1, x_2, x_3, T_1, T_8) = (\sigma_t, \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_3}, \sigma_{T_1}, \sigma_{T_8})(t, x_1, x_2, x_3, T_1, T_8), \qquad \sigma^2 = 1,$$

denote a unit vector orthogonal to a characteristic surface of Eq.(3.1), i.e. a vector tangent locally to a characteristic curve. The characteristic equation for the operator P reads

$$d_R \, (\sigma^2_{x_1} + \sigma^2_{x_2} + \sigma^2_{x_3}) = 0,$$

which implies that $\sigma_{x_k} = 0$, $k = 1, 2, 3$. Thus along each of the characteristic curves the tangent vector has the form $(\sigma_t, 0, 0, 0, \sigma_{T_1}, \sigma_{T_8}))$. This implies that $\mathbf{x} = const$, hence by Eq.(3.1) we conclude that, **on each of the characteristic curves**, the equation

$$\frac{\partial R}{\partial t} = -\frac{\partial}{\partial T_1} \left(\Gamma(T_1) \, R\right) - \frac{\partial}{\partial T_8} \left(B(T_8) \, R\right) \tag{3.3}$$

is satisfied. It follows that the characteristic curve assigned to a point $(\mathbf{x}, T_{10}, T_{80}) \in \mathbb{R}^3 \times \mathbb{R}^2$ is given by the mapping:

$$[0, T] \ni t \mapsto (t, \mathbf{x}, T_1(T_{10}, t), T_8(T_{80}, t)), \tag{3.4}$$

where the functions $(T_1(T_{10}, t), T_8(T_{80}, t))$ are solutions to the initial ode problem:

$$\frac{dT_1}{dt}(t) = \Gamma(T_1), \quad \frac{dT_8}{dt}(t) = B(T_8), \quad T_1(0) = T_{10}, \; T_8(0) = T_{80}. \tag{3.5}$$

As the functions $\Gamma(\cdot)$ and $B(\cdot)$ are independent of $\mathbf{x}$, then the time courses of the functions $T_1(t)$ and $T_8(t)$ are also independent of $\mathbf{x}$.

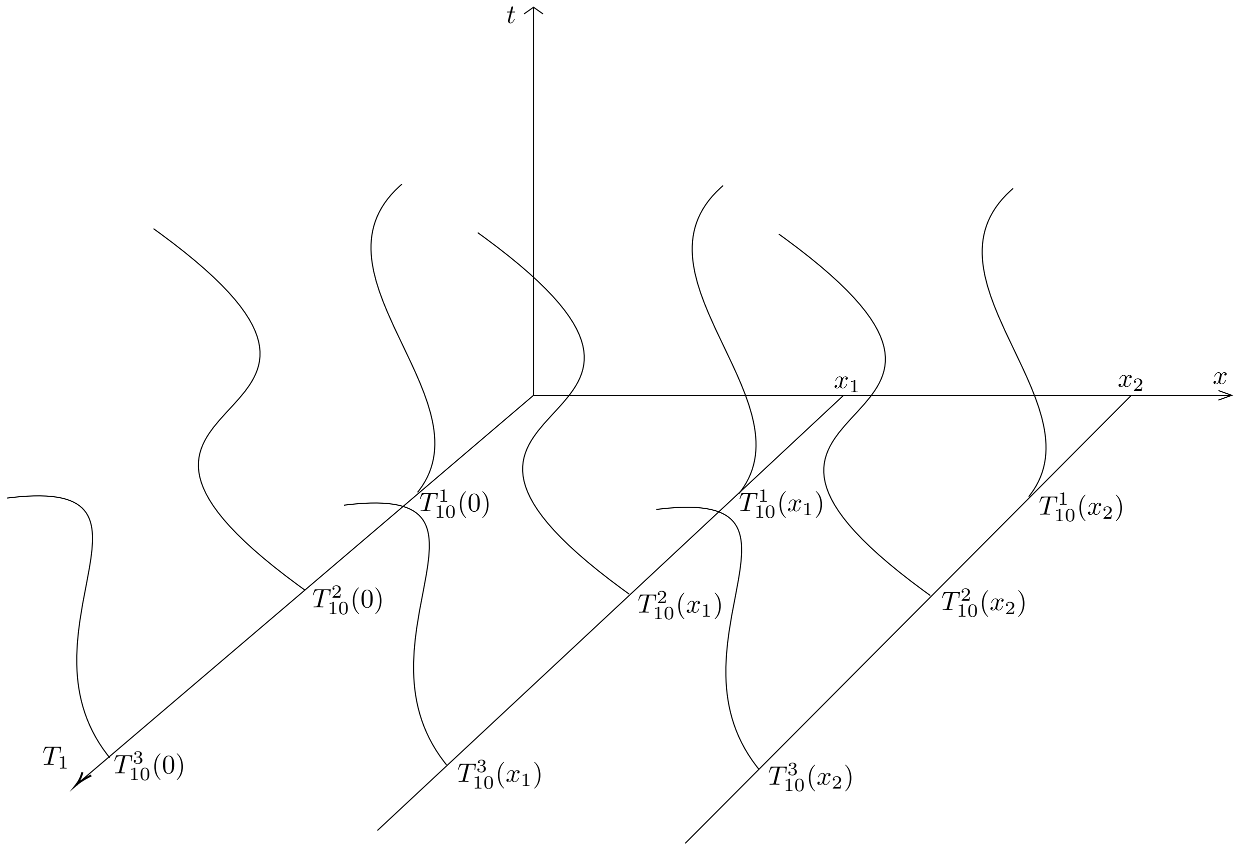

Figure 1: The schematic display of characteristic curves determined by system (3.5). To be able to illustrate properly the features of the problem, we confine ourselves to the case $x \in \mathbb{R}^1$ and consider only T_1-variable instead of (T_1, T_8). Each curve lies in the plane $x = const$. In the picture, there are three groups of curves lying in the planes $x = 0$, $x = x_1$ and $x = x_2$. In the case shown, the characteristic curves depend only on the initial value for $t = 0$ and do not depend on x, i.e. the function Γ depends only on T_1 and not on x. In general, the characteristic curves depend also on the point $\mathbf{x}$ as in Fig. 2.

Remark From what we said above, it follows that we can expect two different ways of information transfer connected with the initial data. Thus, in the $\mathbf{x}$-space, the initial distribution is spread by diffusion, whereas in the space (T_1, T_8) it can be transduced along the projection of the characteristic curves onto the (T_1, T_8)-space. Motivated by this reasoning, we will construct a solution to an initial value problem corresponding to Eq.(3.1). This solution is composed of the heat kernel in $\mathbb{R}^3$ and the curves defined in (3.4). It is given by equality (3.52) in Lemma 3.8 or equality (3.27) in the case of linear Γ and B. $\square$

Remark Although, from the biological point of view, T_1 and T_8 can attain only non-negative values, so in principle Γ and B are defined only on the non-negative half-lines, for technical reasons, we will treat the functions Γ and B as defined on the whole real line. This extension can be done, if these functions are sufficiently smooth. *For simplicity the extended functions, will be denoted in the same way.* $\square$

Assumption 3.2. *Assume that $\Gamma(T_1)$ and $B(T_8)$ are of C^{k+1} class, $k \geq 2$, and that for all (T_{10}, T_{80}) system (3.5) has a unique C^{k+1} solution $(T_1(\cdot), T_8(\cdot))$ satisfying the initial conditions $T_1(0) = T_{10}$, $T_8(0) = T_{80}$, defined for all $t \geq 0$. Suppose that there exists a positive number ρ_{18}, such that*

$$\Gamma(T_1) \geq 0 \quad \textit{for } |T_1| \leq \rho_{18}$$

$$B(T_8) \geq 0 \quad \textit{for } |T_8| \leq \rho_{18}.$$

Remark Below, for simplicity, the symbol $\mathbf{x}$ will be reduced to x. $\square$

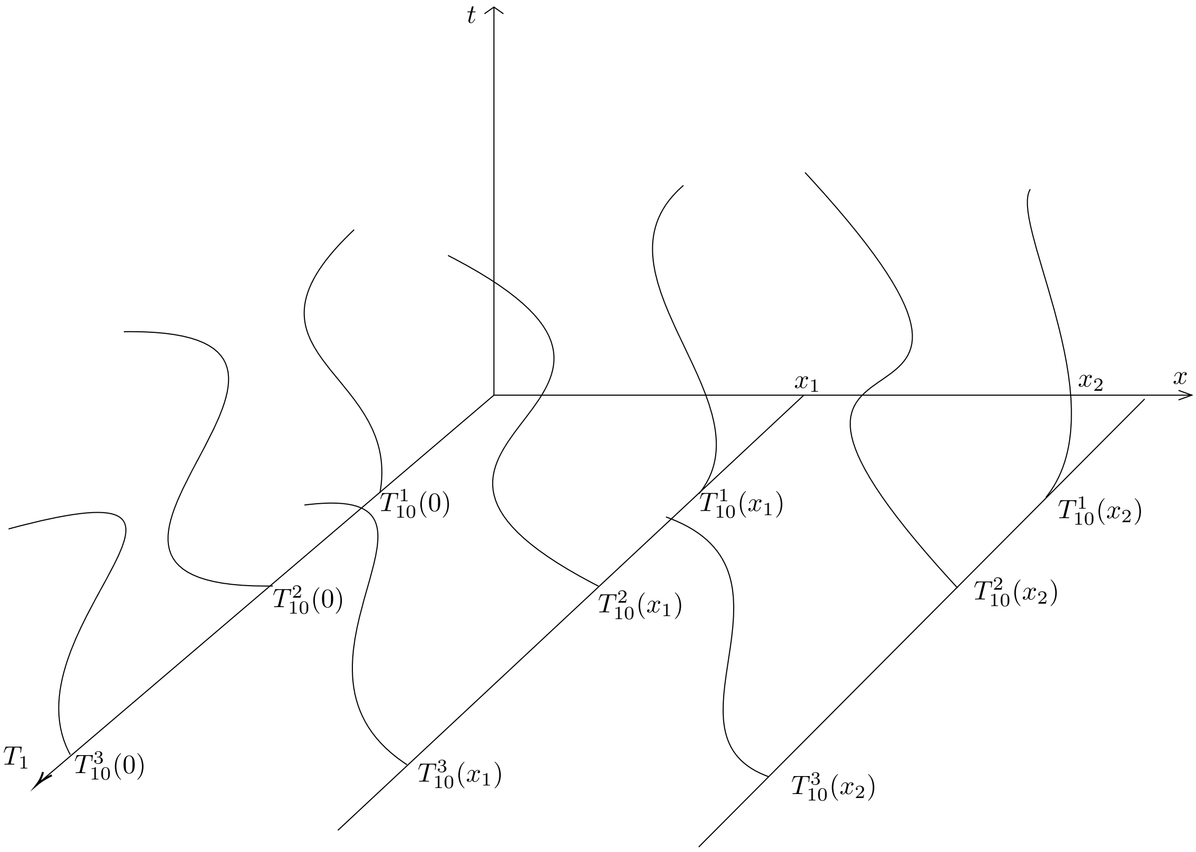

Figure 2: The schematic display of characteristic curves determined by system (3.5). Each curve lies in the plane $x = const$. In the picture, there are three groups of curves lying in the planes $x = 0$, $x = x_1$ and $x = x_2$. In contrast to Fig. 1, the characteristic curves depend also on x. In this case, the function Γ depends not only on T_1 but also on x.

Assumption 3.3. *Assume that for all $x \in \overline{\Omega}$, $R_0(x, T_1, T_8) \not\equiv 0$ only for (T_1, T_8) from some open precompact set in $\mathbb{R}^2_+$.*

The idea applied in this approach is to construct the solution to the boundary initial value problem corresponding to Eq.(3.1) by means of the Green's function of the parabolic part of this equation and the reversed in time solutions to system (3.5).

We have:

$$\frac{dT_1}{dt} = \Gamma(T_1), \quad T_1(0) = T_{10}. \tag{3.6}$$

hence

$$\int_{T_{10}}^{T_1} (\Gamma(s))^{-1} ds = t. \tag{3.7}$$

Thus for $Int(y) := \displaystyle\int_{(\cdot)}^{y} (\Gamma(s))^{-1} ds$, we obtain

$$Int(T_1) - Int(T_{10}) = t.$$

According to Assumption 3.2, given T_{10} we can uniquely determine the value of $T_1(T_{10}, t)$, for any $t \geq 0$. On the other hand, fixing T_1 and $t \geq 0$, we can ask about the initial condition T_{10} such that the value of solution to the initial problem (3.6) at time t is equal to T_1. This initial condition will be denoted below by $T_{10}(T_1, t)$.

It follows by differentiation of (3.7) with respect to t, treating T_1 as given, that $T_{10}(T_1, t)$ is a solution to the initial value problem:

$$\frac{\partial T_{10}}{\partial t} \cdot (\Gamma(T_{10}))^{-1} = -1, \quad T_{10}(T_1, 0) = T_1$$

so

$$\frac{\partial T_{10}}{\partial t} = -\Gamma(T_{10}), \quad T_{10}(T_1, 0) = T_1. \tag{3.8}$$

whereas, for fixed $t \geq 0$, by differentiation of (3.7) with respect to T_1 we obtain:

$$\frac{\partial T_{10}}{\partial T_1} = \frac{\Gamma(T_1)^{-1}}{\Gamma(T_{10})^{-1}} = \frac{\Gamma(T_{10})}{\Gamma(T_1)}, \tag{3.9}$$

which should be written in a detailed form as

$$\frac{\partial T_{10}}{\partial T_1}(t) = \frac{\Gamma(T_1)^{-1}}{\Gamma(T_{10}(T_1,t))^{-1}} = \frac{\Gamma(T_{10}(T_1,t))}{\Gamma(T_1)}. \tag{3.10}$$

In the similar way, we can prove that

$$\frac{\partial T_{80}}{\partial t} = -B(T_{80}), \quad T_{80}(T_8, 0) = T_8. \tag{3.11}$$

and

$$\frac{\partial T_{80}}{\partial T_8}(t) = \frac{B(T_1)^{-1}}{B(T_{80}(T_8,t))^{-1}} = \frac{B(T_{80}(T_8,t))}{B(T_8)}. \tag{3.12}$$

Remark For completeness we derived relation (3.9) explicitly, but it is a special case of [18, Corollary 3.1, Chapter V], according to which

$$\frac{\partial T_1}{\partial T_{10}}(t) = \exp\Big(\int_0^t \frac{\partial \Gamma(T_1(T_{10},s))}{\partial T_1} ds\Big) = \exp\Big(\int_{T_{10}}^{T_1} \frac{\partial \Gamma(T_1(T_{10},s))}{\partial T_1} \Big[\Gamma(T_1(T_{10},s)\Big]^{-1} dT_1\Big)$$

hence at $T_1 = T_1(T_{10}, t)$

$$\frac{\partial T_{10}}{\partial T_1}(t) = \exp\Big(-\int_0^t \frac{\partial \Gamma(T_1(T_{10},s))}{\partial T_1} ds\Big) = \exp\Big(-\int_{T_{10}}^{T_1} \frac{\partial \Gamma(T_1(T_{10},s))}{\partial T_1} \Big[\Gamma(T_1(T_{10},s)\Big]^{-1} dT_1\Big).$$

On the other hand, using [18, Corollary 3.1, Chapter V] to Eq.(3.8), we obtain obvious equivalent expressions:

$$\frac{\partial T_{10}}{\partial T_1}(t) = \exp\Big(-\int_0^t \frac{\partial \Gamma(T_{10}(T_1,\sigma))}{\partial T_{10}} d\sigma\Big) = \exp\Big(-\int_{T_1}^{T_{10}} \frac{\partial \Gamma(T_{10}(T_1,\sigma))}{\partial T_{10}} \Big[-\Gamma(T_{10}(T_1,\sigma))\Big]^{-1} dT_{10}\Big)$$

and at $T_{10} = T_{10}(T_1, t)$

$$\frac{\partial T_1}{\partial T_{10}}(t) = \exp\Big(\int_0^t \frac{\partial \Gamma(T_{10}(T_1,\sigma))}{\partial T_{10}} d\sigma\Big) = \exp\Big(\int_{T_1}^{T_{10}} \frac{\partial \Gamma(T_{10}(T_1,\sigma))}{\partial T_{10}} \Big[-\Gamma(T_{10}(T_1,\sigma))\Big]^{-1} dT_{10}\Big).$$

These identities hold also for the function Γ depending on T_1 and t. To obtain (3.9) (in the case of Γ independent explicitly on t), we use the change of variables $ds = dT\,(\Gamma(T(T_{10},s)))^{-1}$, by which

$$\int_0^t \frac{\partial \Gamma(T_1(T_{10},s))}{\partial T_1} ds = \int_{T_{10}}^{T_1} \frac{\partial\, log(\Gamma(T))}{\partial T} dT.$$

□

Similar remarks concerns the relation (3.12).

Having the family of curves determined by system (3.6), we can define a candidate for a solution to Eq.(3.1) in the case $B \equiv 0$:

$$R(t, x, T_1) = \int_{\mathbb{R}^3} G_\Gamma(t, x; 0, \xi) R_0(\xi, T_{10}(T_1, t)) d\xi \tag{3.13}$$

where $R_0(\cdot,\cdot,\cdot)$ is the initial concentration of the cells,

$$G_\Gamma = \exp\Big(-\int_0^t \frac{\partial\Gamma}{\partial T_1}\Big(T_1(T_{10},\tau)\Big)\,d\tau\Big)\cdot G_0(t,x;0,\xi), \tag{3.14}$$

or, equivalently:

$$G_\Gamma = \exp\Big(\int_0^t \frac{\partial\Gamma}{\partial T_{10}}\Big(T_{10}(T_1,\tau)\Big)\,d\tau\Big)\cdot G_0(t,x;0,\xi),$$

and

$$G_0(t,x;\tau,\xi) = \frac{1}{(4\pi d_R(t-\tau))^{3/2}}\; e^{-\dfrac{|x-\xi|^2}{4d_R(t-\tau)}} \tag{3.15}$$

is the Green's function for the heat equation in $\mathbb{R}^3$.

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R \tag{3.16}$$

In particular, for $t \geq \tau \geq 0$, it satisfies the equation

$$\frac{\partial G}{\partial t} - d_R\nabla^2 G = \delta(t-\tau)\delta(x-\xi). \tag{3.17}$$

Below, we will often use use the following properties of the fundamental solution for the heat equations in $\mathbb{R}^n$.

Lemma 3.4. *Let $n \geq 1$, $\tau \geq 0$ and*

$$G_0^n(t,x;\tau,\xi) = \frac{1}{(4\pi d_H(t-\tau))^{n/2}}\; e^{-\dfrac{|x-\xi|^2}{4D_H(t-\tau)}}. \tag{3.18}$$

Then the following statements hold:

1. *For any $t > \tau$, $x \in \mathbb{R}^n$:*

$$\int_{\mathbb{R}^n} G_0^n(t,x;\tau,\xi)d\xi = 1$$

2. *If $g(\cdot) \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then, for all $x \in \mathbb{R}^n$:*

$$\lim_{t\to\tau} \int_{\mathbb{R}^n} G_0^n(t,x;\tau,\xi)g(\xi)d\xi = g(x).$$

3. *If $g(\cdot) \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then, for all $t > \tau$, the integral*

$$\int_{\mathbb{R}^n} G_0^n(t,x;\tau,\xi)g(\xi)d\xi$$

is a $C^{1,2}$ solution to the homogeneous heat equation

$$\frac{\partial H}{\partial t} = d_H\nabla^2 H$$

with the initial condition $H(\tau,x) = g(x)$.

4. *If $f \in C_{t,x}^{\alpha/2,\alpha}([0,T]\times\mathbb{R}^n) \cap L_x^\infty(\mathbb{R}^n)$, uniformly with respect to $t \in [0,T]$, then the function*

$$H(t,x) = \int_\tau^t \Big(\int_{\mathbb{R}^n} G_0^n(t,x;\sigma,\xi)f(\sigma,\xi)d\xi\Big)d\sigma.$$

is a $C^{1,2}$ solution to the inhomogeneous heat equation

$$\frac{\partial H}{\partial t} = d_H\nabla^2 H + f$$

with the initial condition $H(\tau, x) = 0$.

Proof Points 2 and 3 are stated in Theorem 1 of Section 2.3.1 in [13], whereas point 1 in the preceding lemma. Point 4 is stated in Theorem 2 of Section 2.3.1 in [13]. $\square$

In the simplest possible case, let us assume that

$$\Gamma(T_1) = sT_1 + s_0. \tag{3.19}$$

Then

$$T_1(T_{10}, t) = (T_{10} + \frac{s_0}{s})e^{st} - \frac{s_0}{s}, \quad T_{10}(T_1, t) = (T_1 + \frac{s_0}{s})e^{-st} - \frac{s_0}{s} \tag{3.20}$$

and it is seen that the ratio

$$\frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} = e^{-st}, \tag{3.21}$$

thus *does not depend on* T_1.

According to the assumed linearity of the function $\Gamma(\cdot)$, the equation

$$\frac{dT_1}{dt} = \Gamma(T_1),$$

has a stable positive singular point $(-\frac{s_0}{s})$, if and only if $s < 0$ and $s_0 > 0$, whereas it has an unstable positive singular point if and only if $s > 0$ and $s_0 < 0$. $\square$

As Γ does not depend on x (and t), then

$$\exp\Big(-\int_0^t \frac{\partial \Gamma}{\partial T_1}\Big(T_1(T_{10}, \tau)\Big)\,d\tau\Big) = \exp(-st)$$

so

$$G_\Gamma = \exp(-st) \cdot G_0(t, x; 0, \xi),$$

Thus, in this case, (3.13) takes the form:

$$R(t, x, T_1) = \int_{\mathbb{R}^3} \exp(-st) \cdot G_0(t, x; 0, \xi) R_0(\xi, T_{10}(T_1, t))\, d\xi\,. \tag{3.22}$$

To show that this function satisfies Eq.(3.1), let us note that

$$R \cdot \frac{\partial \Gamma}{\partial T_1}(T_1) = R \cdot s$$

and by (3.9)

$$\begin{aligned}
&\frac{\partial R}{\partial T_1} \cdot \Gamma(T_1) = \left(\int_{\mathbb{R}^3} G_\Gamma(t, x; 0, \xi) R_{0,T_{10}}(\xi, T_{10}(T_1, t))\,d\xi\right) \cdot \frac{\partial T_{10}}{\partial T_1}(T_1, t) \cdot \Gamma(T_1) = \\
&\left(\int_{\mathbb{R}^3} G_\Gamma(t, x; 0, \xi) R_{0,T_{10}}(\xi, T_{10}(T_1, t))\,d\xi\right) \cdot \frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} \cdot \Gamma(T_1) = \\
&\frac{\partial}{\partial T_{10}}\left(\int_{\mathbb{R}^3} G_\Gamma(t, x; 0, \xi) R_0(\xi, T_{10}(T_1, t))\,d\xi\right) \cdot \Gamma(T_{10}(T_1, t)).
\end{aligned}$$

In view of the last equality, calculating the time derivative of the function R defined by (3.22), we obtain

$$\frac{\partial R}{\partial t} = d_R\nabla^2 R - sR + \frac{dT_{10}}{dt}\frac{\partial}{\partial T_{10}}\int_{\mathbb{R}^2} G_\Gamma(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t))d\xi =$$

$$d_R\nabla^2 R - sR - \Gamma(T_{10}(T_1,t))\frac{\partial T_1}{\partial T_{10}}\frac{\partial}{\partial T_1}\int_{\mathbb{R}^2} G_\Gamma(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t))d\xi =$$

$$d_R\nabla^2 R - R\frac{\partial \Gamma(T_1)}{\partial T_1} - \Gamma(T_1)\frac{\partial R}{\partial T_1}.$$

We have thus shown that the function given by (3.22) satisfies Eq.(3.1).

This construction can be generalized to the case of nonzero linear function B. Let us take:

$$B = rT_8 + r_0. \tag{3.23}$$

Then

$$T_8 = (T_{80} + \frac{r_0}{r})e^{rt} - \frac{r_0}{r}, \quad T_{80} = (T_8 + \frac{r_0}{r})e^{-rt} - \frac{r_0}{r}, \tag{3.24}$$

$$\frac{dT_{80}}{dt} = -B(T_{80}), \tag{3.25}$$

and

$$\frac{\partial B}{\partial T_8} = r,$$

so *does not depend on* T_8. Also, the ratio

$$\frac{B(T_{80}(T_8,t))}{B(T_8)} = e^{-rt}, \tag{3.26}$$

thus does not depend on T_8. In this way, repeating the arguments concerning the function Γ, we obtain

$$R(t,x,T_1,T_8) = \int_{\mathbb{R}^3} G_{\Gamma B}(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))d\xi\,. \tag{3.27}$$

with

$$G_{\Gamma B} = \exp(-(s+r)t)\cdot G_0(t,x;0,\xi). \tag{3.28}$$

The function $G_{\Gamma B}$ is a solution of the equation

$$\frac{\partial G}{\partial t} - d_R\nabla^2 G + (s+r)G = \delta(t)\delta(x-\xi). \tag{3.29}$$

Let us note that in the considered linear case, $G_{\Gamma B}$ depends neither on T_1 nor on T_8. For R defined by (3.27), we have, by means of (3.8) and (3.11)

$$\frac{\partial R}{\partial t} = d_R\nabla^2 R - \Big(\Gamma'(T_1) + B'(T_8)\Big)R - \frac{\partial R}{\partial T_1}\cdot\Gamma(T_1(t)) - \frac{\partial R}{\partial T_8}\cdot B(T_8(t)),$$

where

$$\Gamma'(T_1) := \frac{\partial \Gamma(y)}{\partial y}\Big|_{y=T_1} = s, \quad B'(T_8) := \frac{\partial B(y)}{\partial y}\Big|_{y=T_8} = r.$$

As

$$\frac{\partial R}{\partial T_1}\cdot\Gamma(T_1) = \frac{\partial}{\partial T_{10}}\left(\int_{\mathbb{R}^3} G_{\Gamma B}(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))d\xi\right)\cdot\frac{\Gamma(T_{10}(T_1,t))}{\Gamma(T_1)}\,\Gamma(T_1).$$

and

$$\frac{\partial R}{\partial T_8}\cdot B(T_8) = \frac{\partial}{\partial T_{80}}\left(\int_{\mathbb{R}^3} G_{\Gamma B}(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))d\xi\right)\cdot B(T_8)\frac{B(T_{80}(T_8,t))}{B(T_8)}$$

we conclude that R defined by (3.27) satisfies Eq.(3.1).

Before proceeding to nonlinear Γ and B, let us consider a non-homogeneous case. By considering the function $\exp\left((s+r)t\right)u(t,x)$ and using point 4 of Lemma 3.4 with $\tau=0$, we conclude that the solution to the initial value problem

$$\frac{\partial u}{\partial t} = d_R\nabla^2 u - su - ru + f(t,x), \quad u(0,x)=0, \tag{3.30}$$

has the form

$$u(t,x) = \int_0^t \Big(\int_{\mathbb{R}^3} G_{\Gamma B}(t,x;\tau,\xi)f(\tau,\xi)d\xi\Big)d\tau. \tag{3.31}$$

Note that u given by (3.31) does not depend on T_1, T_8 hence $\dfrac{\partial u}{\partial T_1}\equiv 0$, $\dfrac{\partial u}{\partial T_8}\equiv 0$. It follows that

$$\frac{\partial R}{\partial T_1}\Gamma(T_1) + \frac{\partial R}{\partial T_8}B(T_8) = \frac{\partial (R+u)}{\partial T_1}\Gamma(T_1) + \frac{\partial (R+u)}{\partial T_8}B(T_8).$$

In consequence, $R+u$, where R is given by (3.27) and u by (3.31) satisfies the equation

$$\frac{\partial Y}{\partial t} = d_R\nabla^2 Y - \frac{\partial}{\partial T_1}\left(\Gamma(T_1)\,Y\right) - \frac{\partial}{\partial T_8}\left(B(T_8)\,Y\right) + f(t,x) \tag{3.32}$$

with the initial condition $Y(0,x,T_1,T_8) = R_0(0,x,T_1,T_8)$.

Remark Note that fixing T_{10} we have

$$\exp\Big(-\int_0^t \frac{\partial\Gamma}{\partial T_1}\Big(T_1(T_{10},\tau)\Big)\,d\tau\Big) = \frac{\Gamma(T_{10}(T_1,t))}{\Gamma(T_1)}. \tag{3.33}$$

This identity can be proved by the change of integration variables $\tau\mapsto T_1(T_{10},\tau)$ with $dT_1 = \Gamma(T_1)d\tau$, namely

$$\begin{gathered}\int_0^t \frac{\partial\Gamma}{\partial T_1}\Big(T_1(T_{10},\tau)\Big)\,d\tau = \int_{T_{10}}^{T_1(T_{10},t)} \frac{\partial\Gamma}{\partial T_1}\Big(T_1\Big)\Gamma(T_1)^{-1}\,dT_1 = \\ \int_{T_{10}}^{T_1(T_{10},t)} \frac{\partial}{\partial T_1}log\Big(\Gamma(T_1)\Big)\,dT_1 = log\Big(\frac{\Gamma(T_1(T_{10},t))}{\Gamma(T_1(T_{10},0))}\Big) = log\Big(\frac{\Gamma(T_1)}{\Gamma(T_{10}(T_1,t))}\Big)\end{gathered} \tag{3.34}$$

which gives (3.33). Likewise, for fixed T_1, by changing of integration variables $\tau\mapsto T_{10}(T_1,\tau)$ we have

$$\begin{aligned}&\int_0^t \frac{\partial\Gamma}{\partial T_{10}}\Big(T_{10}(T_1,\tau)\Big)d\tau = -\int_{T_1}^{T_{10}(T_1,t)} \frac{\partial}{\partial T_{10}}\Gamma(T_{10})\Big(\frac{dT_{10}}{d\tau}\Big)^{-1}dT_{10} = \\ &-\int_{T_1}^{T_{10}(T_1,t)} (\Gamma(T_{10}))^{-1}\frac{\partial}{\partial T_{10}}\Gamma(T_{10})\;dT_{10} = -\int_{T_1}^{T_{10}(T_1,t)} \frac{\partial}{\partial T_{10}}\log(\Gamma(T_{10}))\,dT_{10} = \\ &-\log\Big(\frac{\Gamma(T_{10}(T_1,t))}{\Gamma(T_1)}\Big) = \log\Big(\frac{\Gamma(T_1)}{\Gamma(T_{10}(T_1,t))}\Big)\end{aligned}$$

hence

$$\exp\Big(-\int_0^t \frac{\partial\Gamma}{\partial T_{10}}\Big(T_{10}(T_1,\tau)\Big)\,d\tau\Big) = \frac{\Gamma(T_{10}(T_1,t))}{\Gamma(T_1)}. \tag{3.35}$$

By using the second equality in (3.20), we check that for $\Gamma(T_1) = sT_1 + s_0$,

$$\frac{\Gamma(T_{10}(T_1,t))}{\Gamma(T_1)} = \exp(-st) \tag{3.36}$$

in agreement with (3.21). Similarly,

$$\exp\Big(-\int_0^t \frac{\partial B}{\partial T_8}\Big(T_8(T_{80},\tau)\Big)\,d\tau\Big) = \frac{B(T_{80}(T_8,t))}{B(T_8)} = \exp\Big(\int_0^t \frac{\partial B}{\partial T_{80}}\Big(T_{80}(T_8,\tau)\Big)\,d\tau\Big). \tag{3.37}$$

Equalities (3.33),(3.35) and (3.37) will be confirmed by the form of the right hand side of (3.52) in Lemma 3.8. □

Let us consider the general form of Γ and B. To this end, let us first set:

$$S = \Gamma B R. \tag{3.38}$$

Then Eq.(3.1) changes to

$$\frac{\partial S}{\partial t} = d_R \nabla^2 S - \Gamma(T_1)\frac{\partial S}{\partial T_1} - B(T_8)\frac{\partial S}{\partial T_8}. \tag{3.39}$$

It seen that the (t, T_1, T_8)-projections of characteristics of the hyperbolic part of Eq.(3.39) are still determined by Eqs (3.5). Hence, as it can be easily checked, the solution to the initial value problem for Eq. (3.39) is given by the formula

$$S(t, x, T_1, T_8) = \int_{\mathbb{R}^3} G_0(t, x; 0, \xi) S_0(\xi, T_{10}(T_1, t), T_{80}(T_8, t)) d\xi\,. \tag{3.40}$$

Instead of proving it explicitly, we will prove in Lemma 3.8 that the function R corresponding to the solution given by (3.52) via the transformation (3.38) satisfies Eq.(3.1). To do this, we need the following auxiliary results.

Lemma 3.5. *Let Assumption 3.2 be satisfied. Suppose that the function $\Gamma(\cdot) : \mathbb{R} \to \mathbb{R}$ is $(k+1)$-times continuously differentiable. Let $k_1 \geq 0$ and $k_2 \geq 0$. Then for all $T_1 \geq 0$ and for all $t > 0$, the function*

$$\mathcal{K}_1(T_1; t) := \frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} \tag{3.41}$$

is continuously differentiable k_1 and k_2 times with respect to t and T_1 respectively, iff $k_1 + s(k_2)(k_2 + 1) \leq k + 1$, where $s(k_2) = 1$, if $k_2 \geq 1$ and $s(0) = 0$.

Proof By (3.9) we have

$$\frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} = \frac{\partial T_{10}}{\partial T_1} \tag{3.42}$$

and

$$\frac{d}{d\tau}\Big(\frac{\partial T_{10}}{\partial T_1}\Big) = \frac{\partial}{\partial T_1}\Big(\frac{dT_{10}}{d\tau}\Big) = -\frac{\partial}{\partial T_1}\Gamma(T_{10}(T_1, \tau)) = -\Big(\frac{\partial}{\partial T_{10}}\Gamma(T_{10}(T_1, \tau))\Big)\Big(\frac{\partial T_{10}}{\partial T_1}\Big) \tag{3.43}$$

Thus in accordance with Remark after (3.9)

$$L_0(T_1, t) := \frac{\partial T_{10}}{\partial T_1}(t) = \exp\Big(-\int_0^t \frac{\partial \Gamma(T_{10}(T_1, \tau))}{\partial T_{10}} d\tau\Big) \tag{3.44}$$

because for $t = 0$ and $\Gamma(T_1) \neq 0$, we have

$$\frac{\partial T_{10}}{\partial T_1}(0) = \Gamma(T_{10}(T_1, 0))\Gamma(T_1)^{-1}\Big|_{T_{10}=T_1} = 1. \tag{3.45}$$

Next, if T_{1*} is such that $\Gamma(T_{1*}) = 0$, then in the representation (3.44), the limit $T_1 \to T_{1*}$ exists for all $t \geq 0$ and has the form

$$\exp\Big(-\int_0^t \frac{\partial \Gamma(T_1)}{\partial T_1} d\tau\Big)\Big|_{T_1=T_{1*}}$$

which in the case of $\Gamma(T_1) = sT_1 + s_0$ is equal to $\exp(-st)$ in accordance with (3.21). (Let us note that applying de l'Hospital rule to find the limit as $T_1 \to T_{1*}$ of $L_0(T_1, t)$ gives no answer, because we obtain the relation $L_0(T_{1*}, t) = 1 \cdot L_0(T_{1*}, t)$.) Thus, by means of $L_0(T_1, t)$, the function (3.41) can be well determined for all T_1 in the considered region.

Now, we can show the differentiability of the ratio $\dfrac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)}$ for finite $t \geq 0$. We have, according to (3.44),

$$
\begin{aligned}
L_1(T_1,t) &:= \frac{\partial}{\partial T_1}\mathcal{K}_1(T_1;t) = \frac{\partial}{\partial T_1}L_0(T_1,t) = \frac{\partial}{\partial T_1}\frac{\Gamma(T_{10}(T_1,t))}{\Gamma(T_1)} = \frac{\partial}{\partial T_1}\exp(-\int_0^t \frac{\partial\Gamma(T_{10}(T_1,s))}{\partial T_{10}}ds) = \\
&\frac{\partial}{\partial T_1}\Big(-\int_0^t \frac{\partial\Gamma(T_{10}(T_1,s))}{\partial T_{10}}ds\Big)\exp(-\int_0^t \frac{\partial\Gamma(T_{10}(T_1,s))}{\partial T_{10}}ds) = \\
&\Big(-\int_0^t \frac{\partial^2\Gamma(T_{10}(T_1,s))}{\partial T_{10}^2}L_0(T_1,s)ds\Big)\exp(-\int_0^t \frac{\partial\Gamma(T_{10}(T_1,s))}{\partial T_{10}}ds),
\end{aligned}
\tag{3.46}
$$

Likewise:

$$
\begin{aligned}
&L_2(T_1,t) := \frac{\partial^2}{\partial T_1^2}\mathcal{K}_1(T_1;t) = \\
&\frac{\partial}{\partial T_1}L_1(T_1,t) = \frac{\partial}{\partial T_1}\Big[\Big(-\int_0^t \frac{\partial^2\Gamma(T_{10}(T_1,s))}{\partial T_{10}^2}L_0(T_1,s)ds\Big)\exp(-\int_0^t \frac{\partial\Gamma(T_{10}(T_1,s))}{\partial T_{10}}ds)\Big] = \\
&\Big(-\int_0^t \frac{\partial^3\Gamma(T_{10}(T_1,s))}{\partial T_{10}^3}L_0(T_1,s)ds\Big)\exp(-\int_0^t \frac{\partial\Gamma(T_{10}(T_1,s))}{\partial T_{10}}ds) + \\
&\Big(-\int_0^t \frac{\partial^2\Gamma(T_{10}(T_1,s))}{\partial T_{10}^2}L_1(T_1,s)ds\Big)\exp(-\int_0^t \frac{\partial\Gamma(T_{10}(T_1,s))}{\partial T_{10}}ds) + \\
&\Big(-\int_0^t \frac{\partial^2\Gamma(T_{10}(T_1,s))}{\partial T_{10}^2}L_0(T_1,s)ds\Big)L_1(T_1,t).
\end{aligned}
\tag{3.47}
$$

It follows that $L_1(T_1,0)=0$, $L_2(T_1,0)=0$ and $L_1(T_1,t)$ together with $L_2(T_1,t)=0$ is bounded as long as $T_{10}(T_1,t)$ is bounded. In general, it is seen that if Γ is $(k+1)$ times continuously differentiable, then

$$
L_k(T_1,t) := \frac{\partial^k}{\partial T_1^k}\mathcal{K}_1(T_1;t)
$$

can be expressed by the derivatives of Γ up till the $(k+1)$-order, and $L_k(T_1,t)$ is bounded as long as $T_{10}(T_1,t)$ is bounded. Now, by (3.43) and (3.44), one can see that

$$
\frac{\partial\mathcal{K}_1}{\partial t}(T_1;t) = -\Big(\frac{\partial}{\partial T_{10}}\Gamma(T_{10}(T_1,t))\Big)L_0(T_1,t),
$$

$$
\begin{aligned}
&\frac{\partial^2\mathcal{K}_1}{\partial t\partial T_1}(T_1;t) = \frac{\partial}{\partial t}L_1(T_1,t) = \\
&L_0(T_1,t)\Big[-L_0(T_1,t)\frac{\partial^2\Gamma(T_{10}(T_1,t))}{\partial T_{10}^2} + \frac{\partial\Gamma(T_{10}(T_1,t))}{\partial T_{10}}\int_0^t \frac{\partial^2\Gamma(T_{10}(T_1,s))}{\partial T_{10}^2}L_0(T_1,s)ds\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2\mathcal{K}_1}{\partial t^2}(T_1;t) &= -\frac{\partial}{\partial t}\Big(L_0(T_1,t)\frac{\partial}{\partial T_{10}}\Gamma(T_{10}(T_1,t))\Big) = \\
&L_0(T_1,t)\Big[\Big(\frac{\partial}{\partial T_{10}}\Gamma(T_{10}(T_1,t))\Big)^2 + \frac{\partial^2\Gamma(T_{10}(T_1,t))}{\partial T_{10}^2}\Gamma(T_{10}(T_1,t))\Big].
\end{aligned}
$$

By induction, we can show that the derivatives of the form

$$
\frac{\partial^k\mathcal{K}_1}{\partial t^{k_1}\partial T_1^{k_2}}(T_1;t)
$$

exist and are bounded, **iff** the function $\Gamma(\cdot)$ is $k_1+s(k_2)(k_2+1)$-times differentiable, where $s(k_2)=1$, if $k_2\geq 1$ and $s(0)=0$. □

In the same way we can prove:

Lemma 3.6. *Let Assumption 3.2 be satisfied. Suppose that the function $B(\cdot) : \mathbb{R} \to \mathbb{R}$ is $(k+1)$-times continuously differentiable. Let $k_1 \geq 0$ and $k_2 \geq 0$. Then for all $T_8 \geq 0$ and for all $t > 0$, the function*

$$\mathcal{K}_8(T_8; t) := \frac{B(T_{80}(T_8, t))}{B(T_8)} \tag{3.48}$$

is continuously differentiable k_1 and k_2 times with respect to t and T_8 respectively, iff $k_1 + s(k_2)(k_2 + 1) \leq k + 1$, where $s(k_2) = 1$, if $k_2 \geq 1$ and $s(0) = 0$.

Finally, the following auxiliary lemma holds.

Lemma 3.7. *For any function $\widetilde{F}(T_1, T_{10}(T_1, t-\tau)) = F(T_{10}(T_1, t-\tau))/\Gamma(T_1)$ we have for all $\tau \in [0, t]$:*

$$\frac{\partial}{\partial t}\widetilde{F}(T_1, T_{10}(T_1, t-\tau)) = -\frac{\partial}{\partial T_1}\Big(\Gamma(T_1)\tilde{F}(T_{10}(T_1, t-\tau)\Big) = -\frac{\partial}{\partial T_1}F(T_{10}(T_1, t-\tau)) \tag{3.49}$$

and, likewise, for any function $\widetilde{H}(T_8, T_{80}(T_8, t-\tau)) = H(T_{80}(T_8, t-\tau))/B(T_8)$,

$$\frac{\partial}{\partial t}\frac{\widetilde{H}(T_{80}(T_8, t-\tau))}{B(T_8)} = -\frac{\partial}{\partial T_8}\Big(B(T_8)\tilde{H}(T_{80}(T_8, t-\tau))\Big) = -\frac{\partial}{\partial T_8}H(T_{80}(T_8, t-\tau)) \tag{3.50}$$

Proof Let us show the first of these equalities. We have

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{F(T_{10}(T_1, t-\tau))}{\Gamma(T_1)} = \frac{1}{\Gamma(T_1)}\frac{\partial}{\partial t}F(T_{10}(T_1, t-\tau)) = \frac{1}{\Gamma(T_1)}\Big[\frac{\partial}{\partial T_{10}}F(T_{10}(T_1, t-\tau))\Big] \cdot \frac{\partial T_{10}(T_1, t-\tau)}{\partial t} = \\
&\frac{1}{\Gamma(T_1)}\Big[\frac{\partial}{\partial T_1}F(T_{10}(T_1, t-\tau))\Big] \cdot \frac{\partial T_1}{\partial T_{10}} \cdot \frac{\partial T_{10}(T_1, t-\tau)}{\partial t} = \\
&-\frac{1}{\Gamma(T_1)} \cdot \frac{\Gamma(T_1)}{\Gamma(T_{10}(T_1, t-\tau))} \cdot \Gamma(T_{10}(T_1, t-\tau))\Big[\frac{\partial}{\partial T_1}F(T_{10}(T_1, t-\tau))\Big] = -\frac{\partial}{\partial T_1}F(T_{10}(T_1, t-\tau))
\end{aligned} \tag{3.51}$$

Thus (3.49) is proved. Similarly we prove the validity of (3.50). □

Lemma 3.8. *The function*

$$R(t, x, T_1, T_8) = \int_{\mathbb{R}^3} G_0(t, x; 0, \xi)\mathcal{K}(T_1, T_8; t)R_0(\xi, T_{10}(T_1, t), T_{80}(T_8, t))d\xi\,, \tag{3.52}$$

where

$$\mathcal{K}(T_1, T_8; t) := \frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)}\frac{B(T_{80}(T_8, t))}{B(T_8)}, \tag{3.53}$$

is a solution to equation

$$\frac{\partial R}{\partial t} = d_R\nabla^2 R - \frac{\partial}{\partial T_1}\left(\Gamma(T_1)\, R\right) - \frac{\partial}{\partial T_8}\left(B(T_8)\, R\right) \tag{3.54}$$

with the initial condition

$$R(0, x, T_1, T_8) = R_0(x, T_1, T_8).$$

If $R_0 \in C^{\alpha,2,2}_{x,T_1,T_8}$, $\alpha \in (0, 1)$, whereas Γ and B are of C^3 class of their arguments in $\mathbb{R}^2_+$, then R given by (3.52) is of $C^2(R^2)$ class with respect to (T_1, T_8) and $C^{1+\alpha/2,2+\alpha}_{t,x}([0, T] \times \mathbb{R}^3)$.

Proof The part of the time derivative of $u(t, x, T_1, T_8)$ taken with respect to t inside G_0 is equal to:

$$\Big(\frac{\partial R}{\partial t}\Big)^0 = \int_{\mathbb{R}^3} G_{0,t}(t, x; 0, \xi)\mathcal{K}(T_1, T_8; t)R_0(\xi, T_{10}(T_1, t), T_{80}(T_8, t))d\xi = d_R\nabla^2 R(t, x, T_1, T_8).$$

Now, let us consider the t-differentiation of the function

$$\mathcal{S} = \mathcal{K}(T_1, T_8; t) R_0(\xi, T_{10}(T_1, t), T_{80}(T_8, t)) =$$

$$\frac{1}{\Gamma(T_1) B(T_8)} \Big[\Gamma(T_{10}(T_1, t)) B(T_{80}(T_8, t)) R_0(\xi, T_{10}(T_1, t), T_{80}(T_8, t)) \Big].$$

We have:

$$\frac{\partial \mathcal{S}}{\partial t} = \frac{B(T_{80}(T_8, t))}{B(T_8)} \Big\{ \frac{\partial}{\partial t} \Big[\frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} R_0(\xi, T_{10}(T_1, t), Y) \Big] \Big\} \Big|_{Y = T_{80}(T_8, t))} +$$
$$\frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} \Big\{ \frac{\partial}{\partial t} \Big[\frac{B(T_{80}(T_8, t))}{B(T_8)} R_0(\xi, Y, T_{80}(T_8, t))) \Big] \Big\} \Big|_{Y = T_{10}(T_1, t)}$$

Thus, due to Lemma 3.7

$$\frac{\partial \mathcal{S}}{\partial t} = -\frac{B(T_{80}(T_8, t))}{B(T_8)} \Big\{ \frac{\partial}{\partial T_1} \Big[\Gamma(T_{10}(T_1, t)) R_0(\xi, T_{10}(T_1, t), Y) \Big] \Big\} \Big|_{Y = T_{80}(T_8, t))} -$$
$$\frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} \Big\{ \frac{\partial}{\partial T_8} \Big[B(T_{80}(T_8, t)) R_0(\xi, Y, T_{80}(T_8, t))) \Big] \Big\} \Big|_{Y = T_{10}(T_1, t)} =$$
$$-\frac{\partial}{\partial T_1} \Big[\frac{B(T_{80}(T_8, t))}{B(T_8)} \Gamma(T_{10}(T_1, t)) R_0(\xi, T_{10}(T_1, t), Y) \Big] -$$
$$\frac{\partial}{\partial T_8} \Big[\frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} B(T_{80}(T_8, t)) R_0(\xi, Y, T_{80}(T_8, t))) \Big] = -\frac{\partial}{\partial T_1} (\Gamma(T_1) \mathcal{S}) - \frac{\partial}{\partial T_8} (B(T_8) \mathcal{S})$$

It follows that

$$\int_{\mathbb{R}^3} G_0(t, x; 0, \xi) \frac{\partial \mathcal{S}}{\partial t} d\xi = \int_{\mathbb{R}^3} G_0(t, x; 0, \xi) \Big\{ -\frac{\partial}{\partial T_1} (\Gamma(T_1) \mathcal{S}) - \frac{\partial}{\partial T_8} (B(T_8) \mathcal{S}) \Big\} d\xi =$$
$$-\frac{\partial}{\partial T_1} \Big\{ (\Gamma(T_1) \int_{\mathbb{R}^3} G_0(t, x; 0, \xi) \mathcal{S}) d\xi \Big\} - \frac{\partial}{\partial T_8} \Big\{ (B(T_8) \int_{\mathbb{R}^3} G_0(t, x; 0, \xi) \mathcal{S}) d\xi \Big\}$$

which proves that the function defined by (3.52) satisfies the homogeneous version of Eq.(3.1). The smoothness properties follow from Lemma 3.5, Lemma 3.6 and the properties of the fundamental solution G_0 (see points 3,4 of Lemma 3.4). □

Now, we will find an expression for the solution to the inhomogeneous equation

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1} \left(\Gamma(T_1)\, R \right) - \frac{\partial}{\partial T_8} \left(B(T_8)\, R \right) + f(t, x, T_1, T_8). \tag{3.55}$$

Lemma 3.9. *The function*

$$u(t, x, T_1, T_8) = \int_0^t \Big(\int_{\mathbb{R}^3} \mathcal{K}(T_1, T_8; t - \tau) G_0(t, x; \tau, \xi) f(\tau, \xi, T_{10}(T_1, t - \tau), T_{80}(T_8, t - \tau)) d\xi \Big) d\tau \tag{3.56}$$

where

$$\mathcal{K}(T_1, T_8; t) := \frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)} \frac{B(T_{80}(T_8, t))}{B(T_8)} \tag{3.57}$$

is a solution to Eq.(3.55) with zero initial condition at $t = 0$. If $f \in C^{\alpha/2, \alpha, 2, 2}_{t, x, T_1, T_8}$, $\alpha \in (0, 1)$, whereas Γ and B are of C^3 class of their arguments (in $\overline{\mathbb{R}^2_+}$), then u given by (3.56) is of $C^2(\overline{\mathbb{R}^2_+})$ class with respect to (T_1, T_8) and $C^{1+\alpha/2, 2+\alpha}_{t,x}([0, T] \times \mathbb{R}^3)$.

Proof The part of the time derivative of $u(t, x, T_1, T_8)$ taken with respect to t inside G_0 and in the upper limit of the integral is, by means of s 2 and 3 Lemma 3.4, equal to:

$$\Big(\frac{\partial u}{\partial t}\Big)^0 = \int_0^t \Big(\int_{\mathbb{R}^3} \mathcal{K}(T_1, T_8; t-\tau) G_{0,t}(t,x;\tau,\xi) f(\tau,\xi,T_{10}(T_1,t-\tau), T_{80}(T_8,t-\tau)) d\xi \Big) d\tau +$$

$$\lim_{\tau\to t} \int_{\mathbb{R}^3} \mathcal{K}(T_1, T_8; t-\tau) G_0(t,x;\tau,\xi) f(\tau,\xi,T_{10}(T_1,t-\tau), T_{80}(T_8,t-\tau)) d\xi$$

$$= d_R \nabla^2 u(t,x,T_1,T_8) + f(t,x,T_1,T_8)$$

because $T_{10}(T_1, t-\tau)|_{\tau=t} = T_1$ and $T_{80}(T_8, t-\tau)|_{\tau=t} = T_8$. Thus $f(\tau,\xi,T_{10}(T_1,t-\tau), T_{80}(T_8,t-\tau))|_{\tau=t} = f(\tau,\xi,T_1,T_8)$ and $\mathcal{K}(T_1,T_8;t-\tau)|_{\tau=t} = 1$.

Now, let us consider the derivative

$$\Big(\mathcal{K}(T_1, T_8; t-\tau) f(\tau,\xi,T_{10}(T_1,t-\tau), T_{80}(T_8,t-\tau))\Big)_{,t}$$

The differentiated expression can be written as $(\Gamma(T_1)B(T_8))^{-1}\Psi(\tau,\xi,T_{10}(T_1,t-\tau), T_{80}(T_8,t-\tau))$. We have:

$$\frac{d}{dt}\Big((\Gamma(T_1)B(T_8))^{-1}\Psi(\tau,\xi,Y_1(t-\tau),Y_8(t-\tau)) \Big) =$$

$$\frac{B(Y_8)}{B(T_8)}\Big|_{Y_8=T_{80}(T_8,t-\tau)} \frac{d}{dt}\Big(\frac{1}{\Gamma(T_1)} \cdot \Gamma(T_{10}(T_1,t-\tau)) f(\tau,\xi,T_{10}(T_1,t-\tau),Y_8)\Big)\Big|_{Y_8=T_{80}(T_8,t-\tau)} +$$

$$\frac{\Gamma(Y_1)}{\Gamma(T_1)}\Big|_{Y_1=T_{10}(T_1,t-\tau)} \frac{d}{dt}\Big(\frac{1}{B(T_8)} \cdot B(T_{80}(T_8,t-\tau)) f(\tau,\xi,Y_1,T_{80}(T_8,t-\tau))\Big)\Big|_{Y_1=T_{10}(T_1,t-\tau)}.$$

Using Lemma 3.7 with

$$F(T_{10}(T_1,t-\tau)) = \Gamma(T_{10}(T_1,t-\tau)) f(\tau,\xi,T_{10}(T_1,t-\tau),Y_8)$$

with Y_8 fixed and

$$H(T_{80}(T_8,t-\tau))) = B(T_{80}(T_8,t-\tau)) f(\tau,\xi,Y_1,T_{80}(T_8,t-\tau))$$

with Y_1 fixed, one notes, as in the proof of Lemma 3.8, that:

$$\int_0^t \int_{\mathbb{R}^3} G_0(t,x;\tau,\xi) \Big(\mathcal{K}(T_1,T_8;t-\tau) f(\tau,\xi,T_{10}(T_1,t-\tau),T_{80}(T_8,t-\tau))\Big)_{,t} d\xi d\tau = -$$

$$\int_0^t \int_{\mathbb{R}^3} \Big[\frac{B(T_{80}(T_8,t-\tau))}{B(T_8)} \frac{\partial}{\partial T_1} F(T_{10}(T_1,t-\tau)) + \frac{\Gamma(T_{10}(T_1,t-\tau))}{\Gamma(T_1)} \frac{\partial}{\partial T_8} H(T_{80}(T_8,t-\tau))\Big] G_0(t,x;\tau,\xi) d\xi d\tau$$

$$= -\frac{\partial}{\partial T_1}\Big(\Gamma(T_1) \int_0^t \int_{\mathbb{R}^3} G_0(t,x;\tau,\xi)\mathcal{K}(T_1,T_8;t-\tau) f(\tau,\xi,T_{10}(T_1,t-\tau),T_{80}(T_8,t-\tau)) d\xi\, d\tau\Big)$$

$$-\frac{\partial}{\partial T_8}\Big(B(T_8) \int_0^t \int_{\mathbb{R}^3} G_0(t,x;\tau,\xi)\mathcal{K}(T_1,T_8;t-\tau) f(\tau,\xi,T_{10}(T_1,t-\tau),T_{80}(T_8,t-\tau)) d\xi\, d\tau\Big) =$$

$$-\frac{\partial}{\partial T_1}(\Gamma(T_1)u) - \frac{\partial}{\partial T_8}(B(T_8)u).$$

The smoothness properties follow from Lemma 3.4 (points 3 and 4) together with Lemma 3.5 and Lemma 3.6. The lemma is proved. □

Remark An important note should be made concerning the construction of the solution. As can be seen from the proof of Lemma 3.8 and Lemma 3.9, it is crucial that in the expression for the solution there is a term $(\Gamma(T_1)B(T_8))^{-1}$. Otherwise, the last expression in the sequence of equalities (3.51) would have a form $[-\Gamma(T_1)\frac{\partial}{\partial T_1}F(T_{10}(T_1,t-\tau))]$, so Eq.(3.1), so could not be written as a derivative with respect to T_1. The same concerns the derivative with respect to T_8. In consequence, Eq.(3.55) could not be fulfilled. □

The last remark touches the problem of uniqueness of solutions.

4 Uniqueness of solutions

In this section, we will present two theorems concerning the uniqueness of solutions to different generalizations equations of Eq.(3.55) **under the assumption that they exist**. In these generalizations, we will assume that the functions Γ and B may additionally depend on t and x. Assuming the dependence on t is justified by the fact that in section 8.4 we show that in the case of the product initial data, the existence of solutions to the homogeneous equation (3.1) is implied by the existence to the assigned hyperbolic equation. At the end of section 8.4, we also give an example of solution to Eq.(3.1) in the case of the general initial data.

In the first uniqueness lemma we will consider the equation:

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1}\left(\Gamma(T_1,t)\, R\right) - \frac{\partial}{\partial T_8}\left(B(T_8,t)\, R\right) + f(t,x,T_1,T_8). \tag{4.1}$$

Similarly to the case of Γ and B not depending explicitly on t, the characteristic curves assigned to the hyperbolic part of Eq.(4.1) are given by the system of odes for $t \in [0,T]$:

$$\frac{dT_1}{dt}(t) = \Gamma(T_1,t), \quad \frac{dT_8}{dt}(t) = B(T_8,t), \quad T_1(0) = T_{10},\ T_8(0) = T_{80}. \tag{4.2}$$

Let $\mathcal{W}_1$ be a space of functions $\{u : [0,T] \times \mathbb{R}^3 \times \overline{\mathbb{R}^2_+} \mapsto \mathbb{R}\}$ satisfying the following conditions:
1. $u_{,t}(t,\cdot,T_1,T_8)$ is bounded in the space $L^2(\mathbb{R}^3)$ uniformly in $(t,T_1,T_8) \in [0,T] \times \overline{\mathbb{R}^2_+}$
2. $\|u\|_{W^{2,2}_x(\mathbb{R}^3)} + \|u\|_{C^1_x(\mathbb{R}^3)} \le c_{21}$ uniformly in $(t,T_1,T_8) \in [0,T] \times \overline{\mathbb{R}^2_+}$
3. the derivatives with respect to x behave like $o(|x|^{-2})$ as $|x| \to \infty$
4. $\|u\|_{C^{1,1}_{T_1,T_8}(\mathbb{R}^2_+)} \le c_*$ uniformly with respect to $(t,x) \in [0,T] \times \mathbb{R}^3$.

The following uniqueness result holds.

Lemma 4.1. *Suppose that the functions Γ and B are of C^2 class of their arguments and that for $t \in [0,T]$ the characteristic curves given by solutions to system (4.2) fill out the whole $\overline{\mathbb{R}^2_+}$ and that for $t \in [0,T]$ the set $\overline{\mathbb{R}^2_+}$ is positively invariant with respect to system (4.2). Then, solutions to Eq.(4.1) are unique in the space $\mathcal{W}_1$ (defined above).*

Proof Suppose that the thesis of the lemma is not true. Let D denote the difference between any of two solutions to Eq.(3.55). We thus have:

$$\frac{\partial D}{\partial t} = d_R \nabla^2 D - \frac{\partial}{\partial T_1}\left(\Gamma(T_1,t)\, D\right) - \frac{\partial}{\partial T_8}\left(B(T_8,t)\, D\right) \tag{4.3}$$

and $D(0,x,T_1,T_8) \equiv 0$. Multiplying Eq.(4.3) by D, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}D^2 = d_R \nabla \cdot \Big(D\,\nabla D\Big) - \frac{1}{2}d_R(\nabla D)^2 - \frac{\partial}{\partial T_1}\left(\Gamma(T_1,t)\, D^2\right) + \frac{1}{2}\Gamma(T_1,t)\frac{\partial}{\partial T_1}D^2 - \frac{\partial}{\partial T_8}\left(B(T_8,t)\, D^2\right) \\ + \frac{1}{2}B(T_8,t)\frac{\partial}{\partial T_8}D^2\end{aligned} \tag{4.4}$$

what can be written as

$$\begin{aligned}&\frac{\partial}{\partial t}D^2 = d_R \nabla \cdot \Big(\nabla D^2\Big) - d_R(\nabla D)^2 \\ &-2\frac{\partial}{\partial T_1}\left(\Gamma(T_1,t)\right) D^2 - \Gamma(T_1,t)\frac{\partial}{\partial T_1}D^2 - 2\frac{\partial}{\partial T_8}\left(B(T_8,t)\right) D^2 - B(T_8,t)\frac{\partial}{\partial T_8}D^2.\end{aligned} \tag{4.5}$$

Integrating, for each (t, T_1, T_8) with respect to x over the whole $\mathbb{R}^3$ (by integrating over the finite radius balls and passing to the limit), using the Gauss-Ostrogradskii theorem and denoting $Q = \int_{\mathbb{R}^3} D^2 dx$, we obtain

$$\frac{\partial}{\partial t} Q = -2\frac{\partial}{\partial T_1}\left(\Gamma(T_1,t)\right)\ Q - \Gamma(T_1,t)\frac{\partial}{\partial T_1}\ Q - 2\frac{\partial}{\partial T_8}\left(B(T_8,t)\right)\ Q - B(T_8,t)\frac{\partial}{\partial T_8}\ Q - \mathcal{G}(D)(t,T_1,T_8), \tag{4.6}$$

where $Q(0, T_1, T_8) \equiv 0$ and $\mathcal{G}(D)$ is a functional, which attains positive values for all $D \not\equiv 0$. Suppose that $D \not\equiv 0$. Then $\mathcal{G}(D)$ can be considered as a **given** function of $(t, T_1, T_8) \in C^0$ such that it is strictly positive. Let us consider an auxiliary equation

$$\frac{\partial}{\partial t}\underline{Q} = 2\frac{\partial}{\partial T_1}\left(\Gamma(T_1,t)\right)\ \underline{Q} - \Gamma(T_1,t)\frac{\partial}{\partial T_1}\ \underline{Q} - 2\frac{\partial}{\partial T_8}\left(B(T_8,t)\right)\ \underline{Q} - B(T_8,t)\frac{\partial}{\partial T_8}\ \underline{Q} \tag{4.7}$$

Using the uniqueness result for hyperbolic equations, we conclude that this equation can be satisfied only for $\underline{Q} = 0$. Now, if $Q \not\equiv 0$, then $\dfrac{\partial Q}{\partial t} > 0$ for some $t > 0$, which leads to a contradiction. To show this, let us note that, according to equalities (21) in [13, 3.2.2], the characteristic curves for Eqs (4.7) and (4.6) are the same, so can be parametrized in the same way. Let us consider an arbitrary characteristic curve starting for $t = 0$ at a point $(T_{10}, T_{80}) \in \overline{\mathbb{R}^2_+}$ (parametrized with time): $t \mapsto (t, T_1(T_{10}, t), T_8(T_8(T_{80}))$. Thus, using the second equation of (21) [13, 3.2.2], we obtain for $t \in (0, T)$:

$$\frac{d}{dt}\underline{Q}(t,T_1(t),T_8(t)) = -2\frac{\partial}{\partial T_1}\left(\Gamma(T_1(t),t)\right)\underline{Q}(t,T_1(t),T_8(t)) - 2\frac{\partial}{\partial T_8}\left(B(T_8(t),t)\right)\underline{Q}(t,T_1(t),T_8(t))$$

and

$$\frac{d}{dt}Q(t,T_1(t),T_8(t)) =$$

$$-2\frac{\partial}{\partial T_1}\left(\Gamma(T_1(t),t)\right)Q(T_1(t),T_8(t),t) - 2\frac{\partial}{\partial T_8}\left(B(T_8(t),t)\right)Q(t,T_1(t),T_8(t)) - \mathcal{G}(D)(t,T_1(t),T_8(t)).$$

Both of these equations are supplemented by the initial condition at $t = 0$ equal to 0, i.e. $\underline{Q}(0, T_{10}, T_{80}) = 0$ and $Q(0, T_{10}, T_{80}) = 0$. Let us define:

$$w(t) := 2\frac{\partial}{\partial T_1}\left(\Gamma(T_1(t))\right) + 2\frac{\partial}{\partial T_8}\left(B(T_8(t))\right)$$

$$\underline{Q}_* := \underline{Q} \cdot \exp(\int_0^t w(s)ds), \quad Q_* := Q \cdot \exp(\int_0^t w(s)ds).$$

In this way, the equations for $\underline{Q}$ and Q along the characteristics can be transformed to:

$$\frac{d}{dt}\underline{Q}_*(t,T_1(t),T_8(t)) = 0$$

and

$$\frac{d}{dt}Q_*(t,T_1(t),T_8(t)) = -\mathcal{G}(t,T_1(t),T_8(t)) \cdot \exp(\int_0^t w(s)ds).$$

It follows that $\underline{Q}_*(t, T_1(t), T_8(t)) = 0$ hence $\underline{Q}(t, T_1(t), T_8(t)) = 0$ for all $t \in [0, T]$. Finally, if $\mathcal{G}(t, T_1(t), T_8(t)) \not\equiv 0$, then $\dfrac{dQ_*}{dt} \le 0$ for $t \in (0, T]$. Thus, as $Q_* \ge 0$, we must have $Q_* \equiv 0$, so $\mathcal{G} \equiv 0$, in contradiction to our assumption. $\square$

In Lemma 4.1 we assumed the integrability of solutions with respect to x, however, for each $x \in \mathbb{R}^3$, we could assume only their boundedness with respect to (T_1, T_8). In contrast, in the next lemma we will only assume integrability with respect to (T_1, T_8). Moreover, in this approach, we are able to consider the dependence of the functions Γ and B on x. Besides, to obtain uniqueness, we do not have to assume anything about the behaviour of the characteristic curves.

Thus, in the next lemma we will consider the equation

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1}\left(\Gamma(T_1, t; x)\, R\right) - \frac{\partial}{\partial T_8}\left(B(T_8, t; x)\, R\right) + f(t, x, T_1, T_8). \tag{4.8}$$

Let $\mathcal{W}_2$ be the space of functions $\{u : [0, T] \times \mathrm{I\!R}^3 \times \overline{\mathrm{I\!R}^2_+} \mapsto \mathrm{I\!R}\}$ satisfying the following conditions:

1. there exists a constant c^* such that

$$\|\|u_{,t}(t, x, \cdot, \cdot)\|_{L^2(\mathrm{I\!R}^2_+)} + \|u(t, x, \cdot, \cdot)\|_{W^{1,2}_{T_1,T_8}(\mathrm{I\!R}^2_+)} + \|u(t, x, \cdot, \cdot)\|_{C^{1,1}_{T_1,T_8}(\mathrm{I\!R}^2_+)} \le c^*$$

uniformly in $(t, x) \in [0, T] \times \mathrm{I\!R}^3$. Moreover, $u = o(|(T_1, T_8)|^{-2})$ as $|(T_1, T_8)| \to \infty$

2. the first and second derivatives of u with respect to the components of x are bounded in the space $L^2(\mathrm{I\!R}^2_+)$ uniformly in $(t, x) \in [0, T] \times \mathrm{I\!R}^3$, i.e. are square-integrable with respect to (T_1, T_8) over $\mathrm{I\!R}^2_+$.

Lemma 4.2. *Suppose that the functions Γ and B are of C^2 class with respect to t, T_1 and T_8 and of C^1 class with respect to x. Then, solutions to Eq.(4.8) are unique in the space $\mathcal{W}_2$.*

Proof The proof will resemble the proof of Lemma 4.1, however this time we will carry out the integration with respect to T_1 and T_8 instead of x. As in the previous case, the difference of solutions D satisfies the equation:

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}D^2 = \frac{1}{2}d_R \nabla \cdot \left(\nabla D^2\right) - \frac{1}{2}d_R (\nabla D)^2 \\ &-\frac{\partial}{\partial T_1}\left(\Gamma(T_1, t; x)\, D^2\right) + \frac{1}{2}\Gamma(T_1, t; x)\frac{\partial}{\partial T_1}D^2 - \frac{\partial}{\partial T_8}\left(B(T_8, t; x)\, D^2\right) + \frac{1}{2}B(T_8, t; x)\frac{\partial}{\partial T_8}D^2.\end{aligned} \tag{4.9}$$

Let us note that

$$-\frac{\partial}{\partial T_1}\left(\Gamma\, D^2\right) + \frac{1}{2}\Gamma\frac{\partial}{\partial T_1}D^2 = -\frac{\partial}{\partial T_1}\left(\Gamma\, D^2\right) + \frac{1}{2}\frac{\partial}{\partial T_1}(\Gamma D^2) - \frac{1}{2}D^2\frac{\partial}{\partial T_1}\Gamma = -\frac{1}{2}\frac{\partial}{\partial T_1}(\Gamma D^2) - \frac{1}{2}D^2\frac{\partial}{\partial T_1}\Gamma.$$

Thus, omitting for brevity the arguments of the functions Γ and B, we obtain

$$\frac{\partial}{\partial t}D^2 = d_R \nabla \cdot \left(\nabla D^2\right) - d_R (\nabla D)^2 - \frac{\partial}{\partial T_1}\left(\Gamma\, D^2\right) - D^2\frac{\partial}{\partial T_1}\Gamma - \frac{\partial}{\partial T_8}\left(B(T_8)\, D^2\right) - D^2\frac{\partial}{\partial T_8}B \tag{4.10}$$

Now, let

$$P(t, x) := \max_{T_1, T_8 \in \overline{\mathrm{I\!R}^2_+}} \Big\{ -\frac{\partial}{\partial T_1}\Gamma(T_1, t; x) - \frac{\partial}{\partial T_8}B(T_8, t; x)\Big\}.$$

It is seen that $P(t, x)$ is a bounded function of $(t, x) \in [0, T] \times \mathrm{I\!R}^3$. Then

$$\frac{\partial}{\partial t}D^2 \le d_R \nabla \cdot \left(\nabla D^2\right) - d_R (\nabla D)^2 - \frac{\partial}{\partial T_1}\left(\Gamma(T_1)\, D^2\right) - \frac{\partial}{\partial T_8}\left(B(T_8)\, D^2\right) + P(t, x)D^2. \tag{4.11}$$

Next, integrating the both sides with respect to T_1 and T_8, over the sets $B^2(0, \rho_{18}) \bigcup \overline{\mathrm{I\!R}^2_+}$ using the assumptions of the lemma and a refinement of Gauss-Ostrogradskii theorem (see auxiliary Lemma 9.3), we obtain, by passing to the limit $\rho_{18} \to \infty$, and denoting $Q = \int_{\mathrm{I\!R}^2_+} D^2 dT_1 dT_8$,

$$\frac{\partial}{\partial t}Q \le d_R \nabla^2 Q + PQ - \mathcal{G}(D)(t, x), \tag{4.12}$$

where $\mathcal{G}(D)(t,x) = d_R \int_{\mathrm{I\!R}^2_+} (\nabla D)^2 dT_1 dT_8$ can be treated as a given function, which is non-negative and equivalent to 0 only, if $D \equiv 0$. Due to the boundedness of the function Q at x-infinity and the boundedness of the function P, we note that the conditions of [30, Theorem 10, Section 6] (Phragmen-Lindelöf principle) are satisfied. It follows that, as $Q(0,x) \equiv 0$, then $Q(t,x) \leq 0$ for $t \in [0,T]$. As $Q \geq 0$ by definition, it follows that $Q \equiv 0$ and $D \equiv 0$. The lemma is proved. □

For the convenience of the reader, we present the Phragmen-Lindelöf principle in the form of [30, Theorem 10, Section 6].

Lemma 4.3. *(***Phragmen-Lindelöf principle***) Let Ω be an unbounded domain in n-dimensional space and let E be the domain $(0,T) \times \Omega$. Suppose that u satisfies $(L+h)[u] \geq 0$ in E with L a uniformly parabolic operator of the form*

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(t,x) \frac{\partial}{\partial x_i} - \frac{\partial}{\partial t}$$

with bounded coefficients and with $h(t,x)$ bounded from above in E. Assume that u satisfies the growth condition

$$\liminf_{r \longrightarrow \infty} e^{-cr^2} \left[\max_{\substack{x_1^2 + x_2^2 + \cdots + x_n^2 = r^2 \\ 0 \leq t \leq T}} u(t,x) \right] \leq 0$$

for some positive constant c. If $u \leq 0$ for $t = 0$ and $u \leq 0$ on $(0,T) \times \partial\Omega$, then $u \leq 0$ in E.

5 Asymptotics of the solution given by Lemmas 3.8 and 3.9

Let us consider the asymptotics of the functions given by Lemmas 3.8 and 3.9. The asymptotics will be understood either with respect to $t >> 1$ or with respect to a parameter scaling the strength of convective (hyperbolic) terms. This parameter will be denoted below by λ.

In this section we will assume the uniqueness of solutions to the system $(\Gamma(T_1), B(T_8)) = (0,0)$ or at least that there is a unique solution to this system in the support of R (with respect to (T_1, T_8))) for all $t \in [0,T]$ and all $x \in \mathrm{I\!R}^3$ (see Assumption 5.3).

Let us consider the volume of the support of the function $R(t,x,T_1,T_8)$ with respect to (T_1,T_8) as a function of time. (Let us emphasize that we consider the case of Γ and B independent of x.) In view of the right hand side of (3.52), we have thus to consider the 2-d volume of the form:

$$\int_{\mathrm{I\!R}^2} \chi_t(T_1, T_8; \xi) dT_1 dT_8,$$

where $\chi_t(T_1,T_8;\xi) = 1$, **iff** $(T_{10}(T_1,t), T_{80}(T_1,t)) \in \mathit{supp}\, R_0(\xi,\cdot,\cdot)$. Fixing t and changing the variables in the above integral:

$$T_1 \to T_{10}(T_1,t) = T_{10}, \quad T_8 \to T_{80}(T_8,t) = T_{80},$$

we obtain

$$\begin{aligned} dT_1 &= dT_{10} \frac{dT_1}{dT_{10}} = dT_{10} \frac{\Gamma(T_1)}{\Gamma(T_{10})}, \\ dT_8 &= dT_{80} \frac{dT_8}{dT_{80}} = dT_{80} \frac{B(T_8)}{B(T_{80})} \end{aligned} \tag{5.1}$$

and, due to Assumptions 3.2 and 3.3,

$$
\begin{aligned}
\int_{\mathbb{R}^2} \chi_t(T_1, T_8; \xi) dT_1 dT_8 = \int_{\mathbb{R}^2} \chi_0(T_{10}, T_{80}; \xi) \det(J(T_1, T_8; t)) dT_{10} dT_{80} \\
= \int_{\mathbb{R}^2_+} \chi_0(T_{10}, T_{80}; \xi) \det(J(T_1, T_8; t)) dT_{10} dT_{80} \\
= \int_{\mathbb{R}^2_+} \chi_0(T_{10}, T_{80}; \xi) \frac{\Gamma(T_1(T_{10}, t))}{\Gamma(T_{10})} \frac{B(T_8(T_{80}, t))}{B(T_{80})} dT_{10} dT_{80}
\end{aligned} \tag{5.2}
$$

where $J(T_1, T_8; t)$ is the Jacobian matrix of the mapping $(T_{10}, T_{80}) \mapsto (T_1(T_{10}, t), T_8(T_{80}, t))$, i.e.

$$
J(T_1, T_8; t) = \begin{pmatrix} \dfrac{\partial T_1(T_{10}, t)}{\partial T_{10}} & 0 \\ 0 & \dfrac{\partial T_8(T_{80}, t)}{\partial T_{80}} \end{pmatrix} = \begin{pmatrix} \dfrac{\Gamma(T_1(T_{10}, t))}{\Gamma(T_{10})} & 0 \\ 0 & \dfrac{B(T_8(T_{80}, t))}{B(T_{80})} \end{pmatrix}. \tag{5.3}
$$

If Γ and B are linear and given by (3.19) and (3.23), then, similarly to (3.36), we can show straightforwardly that $\det(J(t)) = \exp(st)\exp(rt)$. It follows that in this case

$$
\int_{\mathbb{R}^2} \chi_t(T_1, T_8; \xi) dT_1 dT_8 = \exp((s + r)t) \int_{\mathbb{R}^2_+} \chi_0(T_{10}, T_{80}; \xi) dT_{10} dT_{80}
$$

hence for $s, r < 0$ the support of the function R with respect to T_1 and T_8 decreases in volume as $\exp(-(|s| + |r|)t)$ and, in fact, tends to a stable singular point $(-s_0/s, -r_0/r)$.

To consider more general form of the functions Γ and B, let us suppose the following.

Assumption 5.1. *Suppose that for all $\xi \in \mathbb{R}^3$, for all (T_{10}, T_{80}) from some open neighbourhood of $\operatorname{supp} R_0(\xi, \cdot, \cdot)$ in (T_{10}, T_{80}) space, the solutions $(T_1(T_{10}, t), T_8(T_{80}, t)$ to system (3.5) tend, as $t \to \infty$, to a unique attracting stationary point (A_1, A_8) such that $\Gamma(A_1) = 0$, $B(A_8) = 0$.*

Lemma 5.2. *Suppose that Assumption 3.2 and 5.1 are fulfilled. Then, for each $x \in \overline{\Omega}$, the support of $R(t, x, T_1, T_8)$ tends to a point (A_1, A_8) as $t \to \infty$.*

Proof The proof follows from the form of the right hand side of (5.2). Thus for fixed $(A_1, A_8) \neq (T_{10}, T_{80}) \in \operatorname{supp} R_0(x, T_{10}, T_{80})$, $(T_{10}, T_{80}) \neq (A_1, A_8)$,

$$
\det(J(T_1(T_{10}, t), T_8(T_{80}, t); t)) = \frac{\Gamma(T_1(T_{10}, t)}{\Gamma(T_{10})} \frac{B(T_8(T_{80}, t)}{B(T_{80})} \to 0
$$

as $t \to \infty$. $\square$

Remark In Lemma 5.2, instead of the limit $t \to \infty$, similar behaviour is derived with respect to the asymptotics $\lambda|s|, \lambda|r| \to \infty$. $\square$

5.1 Asymptotic weak limit of the terms $\frac{\partial}{\partial T_1}\Gamma(T_1)R$ and $\frac{\partial}{\partial T_8}B(T_8)R$ with R given by Lemma 3.8

By means of (5.2), we can analyse also the asymptotic weak limit of the terms $\dfrac{\partial}{\partial T_1}\Gamma(T_1)R$ and $\dfrac{\partial}{\partial T_8}B(T_8)R$ with R given by the right hand side (3.52).

Suppose that $R_0(x, T_1, T_8)$, for each $x \in \Omega$, has a compact support S^x inside the positive quadrant of the space (T_1, T_8). Suppose also that

$$
0 \leq \sup_{(T_1, T_8) \in S^x} R_0(x, T_1, T_8) \leq \rho^x < \bar{\rho}. \tag{5.4}
$$

Let us multiply the right hand side of (3.52) by a smooth function $\phi(T_1, T_8)$ of compact support inside the non-negative quadrant $\overline{\mathbb{R}^2_+} = \{T_1 \geq 0, T_8 \geq 0\}$. Integrating by parts with respect to T_1 and using (1.18), we obtain:

$$\int_{\mathbb{R}^2_+} \frac{\partial}{\partial T_1}\Big(\Gamma(T_1)R\Big)\,\phi\, dT_1 dT_8 = \int_{\mathbb{R}^2_+} \Gamma(T_1)R\,\frac{\partial}{\partial T_1}\phi\, dT_1 dT_8.$$

Using (3.52), we obtain the estimate

$$\Big|\int_{\mathbb{R}^2_+} \Gamma(T_1)R\,\frac{\partial}{\partial T_1}\phi\, dT_1 dT_8\Big| \leq \phi_1 \int_{\mathbb{R}^3} G_0(t,x;0,\xi)\,\rho^{\xi}\left[\int_{\mathbb{R}^2_+} \Big|\Gamma(T_1)\,\mathcal{K}(T_1,T_8;t)\Big|\,\chi^{\xi}_t(T_1,T_8)\,dT_1 dT_8\right] d\xi,$$

where $\phi_1 := \sup_{T_1,T_8}\Big|\dfrac{\partial\phi}{\partial T_1}\Big|$, $\mathcal{K}(T_1,T_8;t)$ is given by (3.53) and $\chi^x_t(T_1,T_8) = 1$, iff $(T_{10}(T_1,t), T_{80}(T_8,t)) \in S^x$.

Assumption 5.3. *Suppose that there exists a compact set S_M in the space (T_1, T_8) such that $S^x \Subset S_M$ for all $x \in \mathbb{R}^3$, $\displaystyle\int_{S_M} dT_1 dT_8 = V_M$. Suppose that inside S_M there exists a unique singular point $A = (A_1, A_8)$ of system (3.5) which is attractive. Let*

$$\sup_{(T_1,T_8)\in S_M}\Big|T_1 - A_1\Big| \leq d_1, \qquad \sup_{(T_1,T_8)\in S_M}\Big|T_8 - A_8\Big| \leq d_8.$$

To proceed, we will consider times $t >> 1$ or assume that away of the singular points the absolute value of Γ and B are relatively large. Thus, let us rescale the functions Γ and B by writing

$$\Gamma(T_1) = \lambda\tilde{\Gamma}(T_1), \quad B(T_8) = \lambda\tilde{B}(T_8), \tag{5.5}$$

where $\lambda \in (0,\infty)$ will be a parameter at our disposal.

Assumption 5.4. *Suppose that $\|\tilde{\Gamma}(\cdot)\|_{C^1(\mathbb{R})} = b_1$ and $\|\tilde{B}(\cdot)\|_{C^1(\mathbb{R})} = b_8$, where b_1 and b_8 are independent of λ. Assume that inside the set S_M we have, for $s < 0$, $r < 0$,*

$$\begin{aligned}
&\widetilde{\Gamma}(T_1) \leq s(T_1 - A_1),\ \textit{for}\ T_1 - A_1 > 0, \quad \widetilde{\Gamma}(T_1) \geq s(T_1 - A_1),\ \textit{for}\ T_1 - A_1 < 0,\\
&\widetilde{B}(T_8) \leq r(T_8 - A_8),\ \textit{for}\ T_8 - A_8 > 0, \quad \widetilde{B}(T_8) \geq r(T_8 - A_8),\ \textit{for}\ T_8 - A_8 < 0.
\end{aligned}$$

Assumption 5.4 implies that solutions to system (3.5) satisfy the inequalities:

$$|T_1(T_{10},t) - A_1| \leq |T_{10} - A_1|\exp(\lambda st), \quad |T_8(T_{80},t) - A_8| \leq |T_{80} - A_8|\exp(\lambda rt). \tag{5.6}$$

The same assumption implies that

$$|\Gamma(T_1(T_{10},t)| \leq \lambda|s||T_{10} - A_1|\exp(\lambda st), \quad \text{and} \ \ B(T_8(T_{80},t) \leq \lambda|r||T_{80} - A_8|\exp(\lambda rt). \tag{5.7}$$

Note that, according to (5.3) and (3.53),

$$\det J(T_1,T_8;t) = (\mathcal{K}(T_1,T_8;t))^{-1}. \tag{5.8}$$

We thus have

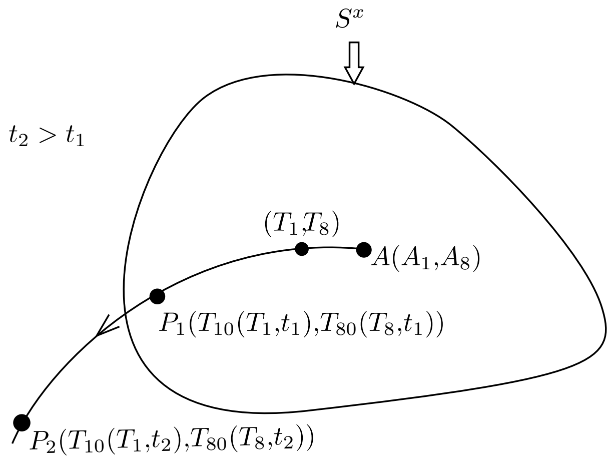

Figure 3: Backward trajectory starting from (T_1, T_8). For t_2 sufficiently large the point (T_1, T_8) escapes from the initial support.

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2_+} \Gamma(T_1)\,\mathcal{K}(T_1, T_8; t)\,\chi^\xi_t(T_1, T_8)\,dT_1 dT_8\Big| = \\
&\Big|\int_{\mathbb{R}^2_+} \Gamma(T_1(T_{10}, t))\det J(T_1(T_{10}, t), T_8(T_{80}, t); t)\,\mathcal{K}(T_1(T_{10}, t), T_8(T_{80}, t); t))\chi^\xi_0(T_{10}, T_{80})\,dT_{10} dT_{80}\Big| = \\
&\Big|\int_{\mathbb{R}^2_+} \Gamma(T_1(T_{10}, t))\chi^\xi_0(T_{10}, T_{80})\,dT_{10} dT_{80}\Big| = \\
&\Big|\int_{\mathbb{R}^2_+} \lambda|s||T_{10} - A_1|\exp(\lambda s t)\,\chi^\xi_0(T_{10}, T_{80})\,dT_{10} dT_{80}\Big| \le \lambda|s|\exp(\lambda s t) d_1 V_M,
\end{aligned} \tag{5.9}$$

hence taking into account the integrability of the fundamental solution G_0, we conclude that

$$\Big|\int_{\mathbb{R}^2_+} \Gamma(T_1) R\,\frac{\partial}{\partial T_1}\phi\,dT_1 dT_8\Big| \le \bar{\rho}\phi_1\lambda|s|\exp(\lambda s t) d_1 V_M \to 0 \tag{5.10}$$

as $\lambda s \to -\infty$ for any finite $t > 0$. Likewise,

$$\Big|\int_{\mathbb{R}^2_+} B(T_8) R\,\frac{\partial}{\partial T_8}\phi\,dT_1 dT_8\Big| \le \bar{\rho}\phi_1\lambda|r|\exp(\lambda r t) d_8 V_M \to 0 \tag{5.11}$$

as $\lambda r \to -\infty$ for any finite $t > 0$. In view of the fact that the function ϕ was arbitrary, it follows that the terms

$$\frac{\partial}{\partial T_1}\Big(\Gamma(T_1) R\Big) \quad \text{and} \quad \frac{\partial}{\partial T_8}\Big(B(T_8) R\Big)$$

vanish **weakly** asymptotically as λ tends to infinity. Similarly, for fixed λ, s and r, the relations (5.10) and (5.11) hold as $t \to \infty$. The last case is shown in Fig. 3.

Now, let us note that, according to (3.52) and (5.8), we obtain as in (5.9):

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} R(t, x, T_1, T_8) dT_1 dT_8 = \int_{\mathbb{R}^3} G_0(t, x; 0, \xi)\Big[\int_{\mathbb{R}^2_+} R_0(\xi, T_{10}, T_{80}) dT_{10} dT_{80}\Big] d\xi =: \\
&\int_{\mathbb{R}^3} G_0(t, x; 0, \xi)\Big[\rho_0(\xi)\Big] d\xi.
\end{aligned}$$

Due to the properties of the function G_0, we infer that, for each $t > 0$ and $x \in \mathbb{R}^3$, we have for nonzero initial data

$$\lim_{\lambda|r|, \lambda|s| \to \infty} \int_{\mathbb{R}^2_+} R(t, x, T_1, T_8) dT_1 dT_8 = \mathcal{R}(t, x) > 0.$$

As for each $(t,x) \in (0,T) \times \mathbb{R}^3$, the volume of the support of $R(t,x,T_1,T_8)$ with respect to (T_1,T_8) tends to 0 (and is concentrated around the point (A_1,A_8)) the following lemma has been shown.

Lemma 5.5. *Suppose that Assumptions 3.2, 3.3, 5.3 and 5.4 are satisfied. Then, for all* $(t,x) \in (0,T] \times \overline{\Omega}$*, as* $\lambda|s| \to \infty$ *and* $\lambda|r| \to \infty$*,*

$$R(t,x,T_1,T_8) \to \mathcal{R}(t,x) \cdot \delta(T_1 - A_1)\delta(T_8 - A_8).$$

5.2 Asymptotic weak limit of the terms $\frac{\partial}{\partial T_1}\Gamma(T_1)R$ and $\frac{\partial}{\partial T_8}B(T_8)R$ with R given by Lemma 3.9

Assumption 5.6. *Suppose that for each* $(t,x) \in [0,T] \times \mathbb{R}^3$ *the support* S^f_{tx} *of the function* $f(t,x,\cdot,\cdot)$ *is a compact set. Let* $S_f = \bigcup_{(x,t)\in[0,T]\times\mathbb{R}^3} S^f_{tx}$*. Suppose that* $\overline{S_f} \subset S^M_f$*, where* S^M_f *is compact and its* 2*-dimensional measure satisfies*

$$|\overline{S^M_f}| < W_M.$$

and that

$$\sup_{t\in[0,T],x\in\mathbb{R}^3,(T_1,T_8)\in\overline{\mathbb{R}^2_+}} |f(t,x,T_1,T_8)| < \mathcal{F} \tag{5.12}$$

for some constants W_M *and* $\mathcal{F}$*.*

Let us note that by changing the integration variable from τ to $\eta = t - \tau$, the right hand side of (3.56) can be written as

$$u(t,x,T_1,T_8) = \int_0^t \Big(\int_{\mathbb{R}^3} \mathcal{K}(T_1,T_8;\eta) G_0(t,x;t-\eta,\xi) f(t-\eta,\xi,T_{10}(T_1,\eta),T_{80}(T_8,\eta)) d\xi \Big) d\eta$$

Proceeding as in the case of the right hand side of (3.52), i.e. multiplying the expression $\dfrac{\partial(B(T_8)u)}{\partial T_8}$, by a function $\phi_1(T_1,T_8)$ of compact support in $\overline{\mathbb{R}^2_+}$, integrating by parts using Assumption 5.6, we obtain:

$$\left| \int_{\mathbb{R}^2_+} \Gamma(T_1) u \frac{\partial}{\partial T_1} \phi \, dT_1 dT_8 \right|$$
$$\leq \mathcal{F}\phi_1 \int_{\mathbb{R}^3} \left\{ \int_0^t G_0(t,x;t-\eta,\xi) \left[\int_{\mathbb{R}^2_+} |\Gamma(T_1)\, \mathcal{K}(T_1,T_8;\eta)|\, \chi^f_\eta(T_1,T_8)\, dT_1 dT_8 \right] d\eta \right\} d\xi,$$

where $\phi_1 =: \sup_{T_1,T_8} \dfrac{\partial \phi}{\partial T_1}$, and

$$\chi^f_t(T_1,T_8) = \begin{cases} 1 & \text{if } (T_{10}(T_1,t), T_{80}(T_8,t)) \in S^f \\ \\ 0 & \text{if } (T_{10}(T_1,t), T_{80}(T_8,t)) \notin S^f \end{cases}$$

Now, we can proceed as in the proof of Lemma 3.8. First, according to (5.7) with t replaced by η, we have

$$\int_{\mathbb{R}^2_+} |\Gamma(T_1)\, \mathcal{K}(T_1,T_8;\eta)|\, \chi^f_t(T_1,T_8)\, dT_1 dT_8 = \int_{\mathbb{R}^2_+} |\Gamma(T_1(T_{10},\eta))| \chi^f_0(T_{10},T_{80})\, dT_{10} dT_{80} \leq$$
$$\int_{\mathbb{R}^2_+} \lambda|s||T_{10} - A_1| \exp(\lambda s \eta)\, \chi^f_0(T_{10},T_{80})\, dT_{10} dT_{80} \Big| \leq \lambda|s| \exp(\lambda s \eta) d_1 W_M < \lambda|s| \exp(\lambda s \eta) d_1 V_M.$$

It follows that for $\Delta\eta = (\sqrt{\lambda})^{-1} << 1$ we have, in view of the properties of the function G_0 (see Lemma 3.4, point 2),

$$\int \Big\{ \int_{\mathbb{R}^3} \Big[\int_0^t \Big(\Gamma(T_1) \mathcal{K}(T_1, T_8; \eta) G_0(t, x; t - \eta, \xi) f(t - \eta, \xi, T_{10}(T_1, \eta), T_{80}(T_8, \eta)) \Big) d\eta \Big] d\xi \Big\} dT_1 dT_8 =$$

$$\int \Big\{ \int_{\mathbb{R}^3} \Big[\int_{\Delta\eta}^t \Big(\Gamma(T_1) \mathcal{K}(T_1, T_8; \eta) G_0(t, x; t - \eta, \xi) f(t - \eta, \xi, T_{10}(T_1, \eta), T_{80}(T_8, \eta)) \Big) d\eta \Big] d\xi \Big\} dT_1 dT_8 +$$

$$\int_0^{\Delta\eta} \Big[\int_{\mathbb{R}^3} \Big(\int \Big\{ \Gamma(T_{10}(T_1, \eta)) G_0(t, x; t - \eta, \xi) f(t - \eta, \xi, T_{10}(T_1, \eta), T_{80}(T_8, \eta)) \Big\} dT_1 dT_8 \Big) d\xi \Big] d\eta =$$

$$O((exp(-\lambda|s|t) - exp(-\Delta\eta\lambda|s|)) +$$

$$\int_0^{\Delta\eta} \Big[\int_{\mathbb{R}^3} \Big(\int \Big\{ \Gamma(T_{10}(T_1, \eta)) G_0(t, x; t - \eta, \xi) f(t - \eta, \xi, T_{10}(T_1, \eta), T_{80}(T_8, \eta)) \Big\} dT_1 dT_8 \Big) d\xi \Big] d\eta \underset{n\to\infty}{\longrightarrow}$$

$$O(exp(-\Delta\eta\lambda|s|)) + W(n),$$

where

$$W(n) = \Big| \lim_{n\to\infty} \int_{\mathbb{R}^3} \Big[\int_{\Delta\eta/n}^{\Delta\eta} \Big(\int \Big\{ \Gamma(T_{10}(T_1, \eta)) G_0(t, x; t-\eta, \xi) f(t-\eta, \xi, T_{10}(T_1, \eta), T_{80}(T_8, \eta)) \Big\} dT_1 dT_8 \Big) d\eta \Big] d\xi \Big|$$

According to the mean value theorem, the integral inside [ ] can be estimated as

$$\lim_{n\to\infty} G_0(t, x; t - \eta_*, \xi) f(t - \eta_*, \xi, T_{10}(T_1, \eta_*), T_{80}(T_8, \eta_*)) \int_{\Delta\eta/n}^{\Delta\eta} \Big((\Gamma(T_1(T_{10}, \eta)) \chi_0^f(T_{10}, T_{80}) \, dT_{10} dT_{80} \Big) d\eta,$$

where $\eta_* \in (\Delta\eta/n, \Delta\eta)$. Using (5.7) we infer that, independently of how small is $\Delta\eta$, we have for $n \to \infty$

$$\int_{\Delta\eta/n}^{\Delta\eta} \Big((\Gamma(T_1(T_{10}, \eta)) \chi_0^f(T_{10}, T_{80}) \, dT_{10} dT_{80} \Big) d\eta \le S^f \cdot C_{T_1} \cdot \int_{\Delta\eta/n}^{\Delta\eta} \lambda |s| \exp(\lambda s t) d\eta \le S^f \cdot C_{T_1},$$

where

$$C_{T_1} = \sup_{T_{10} \in S^M} |T_{10} - A_1|.$$

Using the fact that for any continuous function $\psi(\xi)$, $\int_{\mathbb{R}^3} G_0(t, x; t - \eta_*, \xi) \psi(\xi) d\xi \to \psi(x)$ pointwise as $\eta_* \to 0$, we have

$$\Big| \lim_{\eta_*\to 0} \int_{\mathbb{R}^3} G_0(t, x; t - \eta_*, \xi) f(t - \eta_*, \xi, T_{10}(T_1, \eta_*), T_{80}(T_8, \eta_*)) d\xi \Big| \to f(t, x, T_{10}(T_1, 0), T_{80}(T_8, 0))$$
$$= f(t, x, T_1, T_8).$$

In particular, due to (5.12),

$$lim_{n\to\infty} W(n) \le \mathcal{F} \cdot C_{T_1}.$$

Similar estimates can be obtained for the weak limit of the expression

$$\frac{\partial}{\partial T_8}(B(T_8)u).$$

We can thus conclude the validity of the following lemma.

Lemma 5.7. *Suppose that Assumptions 3.2, 3.3, 5.3, 5.4 and 5.6 are satisfied. Then, asymptotically as $\lambda \to \infty$, the weak limit of the expression*

$$[\frac{\partial}{\partial T_1}(\Gamma(T_1) u(t, x, T_1, T_8))] + [\frac{\partial}{\partial T_8}(B(T_8) u(t, x, T_1, T_8))]$$

at time $t > 0$, with u determined by (3.56), does depend only on the value of the function f at time t and does not depend on the values of this function at smaller times $\tau \in [0, t)$.

6 Weak formulation of Eq.(3.1)

As above, we consider the case $\Omega = \mathbb{R}^3$. In the previous section, we proved that in some sense, the solution $R(t, x, T_1, T_8)$ of (3.1) may converge to a generalized function $\mathcal{R}(t, x) \cdot \delta(T_1 - A_1)\delta(T_8 - A_8)$. In fact, we may consider the weak version of Eq. (3.1) by looking for solutions in the space of distributions $\mathcal{D}'((0, \infty) \times \Omega \times \mathbb{R}^2_+)$.

Lemma 6.1. *Suppose $\mathcal{R}(t, x)$ is a classical solution of the equation $R_t = d_R \nabla^2 R$. Suppose that the functions $\Gamma(T_1)$ and $B(T_8)$ have isolated zeros A_1 and A_8. Then $R(t, x, T_1, T_8) = \mathcal{R}(t, x) \cdot \delta(T_1 - A_1)\delta(T_8 - A_8)$ is a solution to (3.1) in the space $C^{1,2}_{t,x}([0, T) \times \Omega) \cap \times \mathcal{D}'(\mathbb{R}^2_+)$.*

Proof Let us write Eq.(3.1) in the form

$$\frac{\partial R}{\partial t} - d_R \nabla^2 R = -\frac{\partial}{\partial T_1} \left(\Gamma(T_1)\, R\right) - \frac{\partial}{\partial T_8} \left(B(T_8)\, R\right).$$

Multiplying the both sides of this equation by a test function $\psi(T_1, T_8) \in \mathcal{D}(\mathbb{R}^2_+)$, taking $R(t, x, T_1, T_8) = \mathcal{R}(t, x) \cdot \delta(T_1 - A_1)\delta(T_8 - A_8)$ and integrating over $\mathbb{R}^2_+$, we conclude that the left hand side becomes equal to 0, whereas the right hand side, in view of (1.18) is equal to:

$$\begin{aligned} &\langle -\frac{\partial}{\partial T_1} \left(\Gamma(T_1)\, R\right) - \frac{\partial}{\partial T_8} \left(B(T_8)\, R\right), \psi\rangle = \\ &\mathcal{R}(t, x) \cdot \Big(\int_{\mathbb{R}^2_+} \Gamma(T_1)\, \delta(T_1 - A_1)\delta(T_8 - A_8) \cdot \psi_{T_1}(T_1, T_8) dT_1 dT_8 + \\ &\qquad \int_{\mathbb{R}^2_+} B(T_8)\, \delta(T_1 - A_1)\delta(T_8 - A_8) \cdot \psi_{T_8}(T_1, T_8) dT_1 dT_8 \Big) = 0. \end{aligned} \tag{6.1}$$

This proves the lemma. □

Lemma 6.2. *Suppose that Assumptions (5.3) and (5.4) hold. Let $R(t, x, T_1, T_8)$ be a nonegative solution of (3.1) for initial data $R_0(t, x, T_1, T_8)$ as given in (5.4) and Assumption 5.3. Let*

$$\mathcal{R}(t, x) = \int_{\mathbb{R}^2_+} R(t, x, T_1, T_8) dT_1 dT_8.$$

Suppose also that for all $t \in [0, T]$ the 'total mass' integral is finite, i.e.

$$\int_{\mathbb{R}^3} \mathcal{R}(t, x) dx = M_t < \infty.$$

Then

$$R(t, x, T_1, T_8) - \mathcal{R}(t, x)\delta(T_1 - A_1)\, \delta(T_8 - A_8) \to 0$$

in the weak sense in $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^2_+)$ as $t \to \infty$ (for fixed λ) or $\lambda \to \infty$ (for fixed $t > 0$).

Proof For a test function $\psi(x, T_1, T_8) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^2_+)$ of compact support, we have by means of (3.52), (3.53) and (5.3):

$$\begin{aligned} &D_\delta = \langle R(t, x, T_1, T_8) - \mathcal{R}(t, x)\delta(T_1 - A_1)\, \delta(T_8 - A_8), \psi\rangle \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^2_+} R(t, x, T_1, T_8) \left(\psi(x, T_1, T_8) - \psi(x, A_1, A_8)\right) dT_1\, dT_8\, dx = \\ &\int_{\mathbb{R}^3} \int_{\mathbb{R}^2_+} \Big[\int_{\mathbb{R}^3} G_0(t, x; 0, \xi) R_0(\xi, T_{10}(T_1, t), T_{80}(T_8, t)) d\xi \Big] \times \\ &\qquad \Big(\psi(x, T_1, T_8) - \psi(x, A_1, A_8)\Big) \mathcal{K}(T_1, T_8; t)\, dT_1\, dT_8\, dx\,. \end{aligned} \tag{6.2}$$

Now, using Assumption 5.3 we have

$$
\begin{aligned}
&|D_\delta| = \Big| \int_{\mathbb{R}^n} \int_{S_M} \Big[\int_{\mathrm{I\!R}^3} G_0(t,x;0,\xi) R_0(\xi, T_{10}, T_{80}) d\xi \Big] \times \\
&\Big(\psi(x, T_1(T_{10},t), T_8(T_{80},t)) - \psi(x, A_1, A_8)\Big) \mathcal{K}(T_1, T_8; t) \cdot \det(J(T_1, T_8; t)\, dT_{10}\, dT_{80}\, dx \Big| \le \\
&\sup_{(T_{10},T_{80})\in S_M, x\in \mathrm{I\!R}^3} \Big|\psi(x, T_1(T_{10},t), T_8(T_{80},t)) - \psi(x, A_1, A_8)\Big| \times \\
&\qquad \int_{\mathbb{R}^3} \int_{S_M} \Big[\int_{\mathrm{I\!R}^3} G_0(t,x;0,\xi) R_0(\xi, T_{10}, T_{80}) d\xi \Big] dT_{10}\, dT_{80}\, dx.
\end{aligned}
\tag{6.3}
$$

Note that by Assumption 5.4 we have

$$
\begin{aligned}
&|\psi(x, T_1(T_{10},t), T_8(T_{80},t)) - \psi(x, A_1, A_8))| \\
&\qquad \le \|\nabla \psi\|_\infty (|T_{10} - A_1| \exp(\lambda s t) + |T_{80} - A_8| \exp(\lambda r t)).
\end{aligned}
$$

In addition, one computes, by means of (3.52), (5.1) and (5.8),

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \int_{S_M} \Big[\int_{\mathrm{I\!R}^3} G_0(t,x;0,\xi) R_0(\xi, T_{10}, T_{80}) d\xi \Big] dT_{10}\, dT_{80}\, dx = \\
&\int_{\mathbb{R}^3} \int_{S_M} \Big[\int_{\mathrm{I\!R}^3} G_0(t,x;0,\xi) R_0(\xi, T_{10}(T_1,t), T_{80}(T_8,t)) \Big(\det(J(T_1, T_8; t)\Big)^{-1} d\xi \Big] dT_1\, dT_8\, dx = \\
&\int_{\mathbb{R}^3} \int_{S_M} \Big[\int_{\mathrm{I\!R}^3} G_0(t,x;0,\xi) R_0(\xi, T_{10}(T_1,t), T_{80}(T_8,t))\, \mathcal{K}(T_1, T_8; t) d\xi \Big] dT_1\, dT_8\, dx = \\
&\int_{\mathbb{R}^3} \mathcal{R}(t,x) dx = M_t < \infty.
\end{aligned}
$$

Thus putting everything together, we obtain

$$
\begin{aligned}
&|\langle R(t,x,T_1,T_8) - \mathcal{R}(t,x)\delta(T_1 - A_1)\, \delta(T_8 - A_8), \psi\rangle| \le \|\nabla\psi\|_\infty\, M_t \times \\
&\qquad (\sup_{(T_{10},T_{80})\in S_M} |T_{10} - A_1| e^{\lambda s t} + \sup_{(T_{10},T_{80})\in S_M} |T_{80} - A_8| e^{\lambda r t}).
\end{aligned}
\tag{6.4}
$$

This expression clearly converges to zero as $t \to \infty$ (for fixed λ) or $\lambda \to \infty$ (for fixed t). □

Remark In the above proof we did not take into account that ψ has a compact support with respect to x. If this fact is taken into account, the lemma could be proved without the assumption of compactness of R_0 with respect to ξ, because the integral $\int_{\mathrm{I\!R}^3} \int_{S_M} \Big[\int_{\mathrm{I\!R}^3} G_0(t,x;0,\xi) R_0(\xi, T_{10}, T_{80}) d\xi \Big] dT_1 dT_8 dx$ can be replaced by the integral $\int_{S_\psi^x} \int_{S_M} \Big[\int_{\mathrm{I\!R}^3} G_0(t,x;0,\xi) R_0(\xi, T_{10}, T_{80}) d\xi \Big] dT_1 dT_8 dx$, where S_ψ^x denotes the support of ψ with respect to x. This integral is finite. □

7 Integral equality satisfied by the function given by (3.52)

In this section we establish a conservation equality satisfied by the function determined by the right hand side of Eq.(3.52), similarly to Lemma 3.1. Next, we demonstrate that the integral of this function with respect to T_1 and T_8 is equal to a product of a constant and the solution $\mathcal{R} : [0,T] \times \mathrm{I\!R}^3 \mapsto \mathrm{I\!R}$ to the heat equation $\mathcal{R}_t = d_R \nabla^2 \mathcal{R}$. We thus show that the function defined by (3.52) satisfies a necessary condition of being a solution to Eq.(3.1). This property is shown in Lemma 7.2.

Lemma 7.1. *Suppose that the functions Γ, B and R_0 are of C^2 class of their arguments. Suppose that the support of R_0 is compact, both with respect to $(T_1, T_8) \in \overline{\mathrm{I\!R}_+^2}$ and $x \in \mathrm{I\!R}^3$. Then the right hand side of equality (3.52) satisfies the conservation law*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^2_+}\Big[\int_{\mathbb{R}^3} G_0(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))\,\mathcal{K}(T_1,T_8;t)d\xi\Big]\,dT_1\,dT_8\,dx = M_0$$

where M_0 is a constant independent of $t\in[0,T]$.

Proof It suffices to show that the assumptions of Lemma 3.1 are fulfilled. The proof follows from the properties of the fundamental solution $G_0(t,x;0,\xi)$. Let $S^*_\xi = \overline{\bigcup_{T_1,T_8} Supp_\xi(T_1,T_8)}$, where $Supp_\xi(T_1,T_8)$ denotes the support of R_0 with respect to ξ for $(T_1,T_8)\in\mathbb{R}^2_+$. Note that differentiation with respect to the components of x can be replaced by differentiation with respect to corresponding components of ξ, hence

$$\begin{aligned}\nabla_x\Big[\int_{\mathbb{R}^3} G_0(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))\,\mathcal{K}(T_1,T_8;t)d\xi\Big] =\\ -\int_{S^*_\xi}\nabla_\xi G_0(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))\,\mathcal{K}(T_1,T_8;t)d\xi\\ -\int_{S^*_\xi} G_0(t,x;0,\xi)\nabla_\xi R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))\,\mathcal{K}(T_1,T_8;t)d\xi.\end{aligned}$$

Thus if R_0 is of C^1 class with respect to ξ, then, in view of (3.15) with $\tau=0$, $\|\nabla_x R(t,x,T_1,T_8)\|$, vanishes as fast as $\exp(-d_x^2/(4d_Rt))$, where d_x denotes the distance of x from the set S^*_ξ. In view of Lemma 3.1, the lemma is proved. □

Lemma 7.2. *Suppose that $R(t,x,T_1,T_8)$ satisfies Eq.(3.1). Suppose that for all $t\in[0,T]$ the support of $R(t,x,T_1,T_8)$ is compact with respect to (T_1,T_8), i.e. $R(t,x,T_1,T_8)\equiv 0$ if $|T_1|+|T_8|$ is sufficiently large (independently of $t\in[0,T]$ and $x\in\mathbb{R}^3$). Then the function*

$$\mathcal{R}(t,x)=\int_{\mathbb{R}^2_+} R(t,x,T_1,T_8)dT_1dT_8$$

satisfies the diffusion equation

$$\frac{\partial\mathcal{R}}{\partial t}=d_R\nabla^2\mathcal{R}.$$

Proof The proof follows by considering the improper integral over $\mathbb{R}^2_+$ of the both sides of Eq.(3.1) as a limit of integrals over the sets $B(0,r)\cap\mathbb{R}^2_+$. Proceeding as in the proof of Lemma 3.1), that is to say, writing the sum of the last two terms of Eq.(3.1) as $[-\nabla\cdot(\Gamma(T_1)R,B(T_8)R)]$. and using the the Gauss-Ostrogradskii theorem, we conclude that the integral of this sum vanishes. □

From Lemma 7.2, we conclude that if the function defined by the right hand side of Eq.(3.52) is in fact a solution to Eq.(3.1), then it should satisfy the identity

$$\int_{\mathbb{R}^2_+}\Big[\int_{\mathbb{R}^3} G_0(t,x;0,\xi)R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))\,\mathcal{K}(T_1,T_8;t)d\xi\Big]dT_1dT_8 = C\cdot\int_{\mathbb{R}^3} G_0(t,x;0,\xi)\mathcal{R}_0(\xi)d\xi$$

for some function $\mathcal{R}_0$ independent of t and (T_1,T_8). To show this, let us note that due to (5.8) we have

$$\int_{\mathbb{R}^2_+} R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t))\,\mathcal{K}(T_1,T_8;t)dT_1dT_8 = \int_{\mathbb{R}^2_+} R_0(\xi,T_{10},T_{80})\,dT_{10}dT_{80} := \mathcal{R}_0(\xi).$$

8 The case of the product initial data

In this section, we will consider the specific case of the initial data, which can be expressed as a product of the functions depending on x and (T_1,T_8). In this case, in principle, we can give explicit expressions for solutions also in for Eq.(3.1) with the functions Γ and B depending explicitly on t. This section can serve as a test for the validity of Lemma 3.8.

We will thus consider the equation:

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1} \left(\Gamma(T_1, t)\, R \right) - \frac{\partial}{\partial T_8} \left(B(T_8, t)\, R \right). \tag{8.1}$$

Let us seek the function R satisfying Eq.(3.1) in the form of the product

$$R(t, x, T_1, T_8) := R_p(t, x) \cdot R_h(t, T_1, T_8)$$

where the function R_p satisfies the associated heat equation, i.e.

$$\frac{\partial R_p}{\partial t} = d_R \nabla^2 R_p \tag{8.2}$$

and R_h satisfies the hyperbolic equation (3.3), i.e.

$$\frac{\partial R_h}{\partial t} = -\frac{\partial}{\partial T_1}(\Gamma(T_1, t) R_h) - \frac{\partial}{\partial T_8}(B(T_8, t) R_h) \tag{8.3}$$

As, by assumption, R_p does not depend on T_1 and T_8 and R_h does not depend on x, then by calculating the partial derivative, we obtain:

$$\frac{\partial (R_p R_h)}{\partial t} = R_h \nabla_x^2 (R_p) - R_p \cdot \Big(\frac{\partial}{\partial T_1}(\Gamma R_h) + \frac{\partial}{\partial T_8}(B R_h) \Big) =$$
$$\nabla_x^2 (R_p R_h) - \Big(\frac{\partial}{\partial T_1}(\Gamma R_p R_h) + \frac{\partial}{\partial T_8}(B R_p R_h) \Big) = 0.$$

Thus, if the initial data for R have a product form, namely

$$R_0(0, x, T_1, T_8) = R_{p0}(x) \cdot R_{h0}(T_1, T_8) \tag{8.4}$$

then $R_p \cdot R_h$ satisfies Eq.(1.11). We have thus shown the following lemma.

Lemma 8.1. *Suppose that the initial data satisfy condition (8.4) and that the functions Γ and B do not depend on t and x. Then there exists a solution to Eq.(8.1) having the form $R(t, x, T_1, T_8) = R_p(t, x) \cdot R_h(t, T_1, T_8)$, where R_p satisfies Eq.(8.2) and R_h satisfies Eq.(8.3).*

In the simple example, let us consider the case of linear Γ and B functions, which seems to be the simplest example expressing the characteristic features of the analysed equation. First, let us solve the hyperbolic counterpart of Eq.(3.1), i.e. Eq.(3.3). To begin with, note that in the linear case defined by (3.19) and (3.23), Eq.(3.3) takes the form

$$\frac{\partial R_h}{\partial t} = -(s + r) R_h - \Gamma(T_1)\, \frac{\partial}{\partial T_1}\, R_h - B(T_8) \frac{\partial}{\partial T_8}\, R_h \tag{8.5}$$

hence by defining

$$\hat{R}_h := \exp((s + r)t) \cdot R_h. \tag{8.6}$$

we obtain

$$\frac{\partial \hat{R}_h}{\partial t} = -\Gamma(T_1)\, \frac{\partial}{\partial T_1}\, \hat{R}_h - B(T_8) \frac{\partial}{\partial T_8}\, \hat{R}_h \tag{8.7}$$

To simplify the example as much as possible, let us suppose that

$$s = r, \quad s_0 = r_0. \tag{8.8}$$

According to equalities (17) in [13, 3.2.2], the function $\hat{R}_h$ is invariant along the trajectories of the associated flow, i.e.

$$\frac{d\hat{R}_h}{dt} = 0$$

from where it follows that

$$\hat{R}(t, T_1(T_{10}, t), T_8(T_{80}, t)) = \hat{R}_0(T_{10}, T_{80})$$

hence

$$\hat{R}_h(t, T_1, T_8) = \hat{R}_{h0}(T_{10}(T_1, t), T_{80}(T_1, t)),$$

where the functions $T_{10}(T_1, t)$ and $T_{80}(T_1, t)$ are determined by (3.8) and (3.11). Now, inverting (8.6), we obtain:

$$R_h(t, T_1, T_8) = \exp(-(s+r)t)\hat{R}_h(t, T_1, T_8) = \exp(-(s+r)t)\hat{R}_{h0}(T_{10}(T_1, t), T_{80}(T_1, t)) =$$

$$\exp(-(s+r)t)R_{h0}(T_{10}(T_1, t), T_{80}(T_1, t)).$$

Suppose now that the condition (8.4) is satisfied. Then, according to Lemma 8.1, the solution has the form

$$R_p(t, x) \cdot \exp(-(s+r)t)R_{h0}(T_{10}(T_1, t), T_{80}(T_1, t)). \tag{8.9}$$

Let us compare this expression with the equality (3.27), i.e.

$$R(t, x, T_1, T_8) = \int_{\mathbb{R}^3} \exp(-(s+r)t) \cdot G_0(t, x; 0, \xi)R_0(\xi, T_{10}(T_1, t), T_{80}(T_8, t))d\xi =$$

$$\Big(\int_{\mathbb{R}^3} G_0(t, x; 0, \xi)R_{p0}(\xi)d\xi\Big) \cdot \exp(-(s+r)t)R_{h0}(T_{10}(T_1, t), T_{80}(T_8, t)) = R_p(t, x) \cdot R_h(t, T_1, T_8),$$

where we used point 3. of Lemma 3.4. Thus, in the considered case Lemma 8.1 and formula (3.27) give the same results.

Suppose that the support of the initial distribution is equal to the circle $C_{S0} = \overline{B^2((-s_0/s, -r_0/r), p_0)}$. Thus, let us assume that, for $p_0 < -s_0/s$, $\hat{R}_{h0}(T_1, T_8) = cos^4\Big((\pi\sqrt{(T_1 + s_0/s)^2 + (T_8 + r_0/r)^2}\,(2p_0)^{-1}\Big)$ for $(T_1, T_8) \in C_{S0}$ and identically equal to 0 otherwise. For simplicity, let us suppose that $r = s$ and $r_0 = s_0$. The projections of the characteristic curves of Eq.(8.7) on the (T_1, T_8) space, determined by system (3.5), are straight half-lines originating from the point $(T_1, T_8) = (-\frac{s_0}{s}, -\frac{s_0}{s})$. According to (3.20), (3.26), we have

$$(T_{10}(T_1, t) + s_0/s) = (T_1 + s_0/s)\exp(-st), \quad \text{and} \quad (T_{80}(T_8, t) + s_0/s) = (T_8 + s/s_0)\exp(-st). \tag{8.10}$$

In the course of time, the support of the the function $R_h(t, T_1, T_8)$ (corresponding to the support of the function $R_{h0}(T_{10}(T_1, t), T_{80}(T_8, t)))$ changes to a closure of the ball $B^2((-s_0/s, -r_0/r), p(t)) =: C_{St}$, where $p(t) = p_0 \cdot \exp(st)$. It follows that the area of the support behaves as $\pi p_0^2 \exp(2st)$. On the other hand, using (8.10) and the fact that $2\pi\int_0^1 cos^4(\pi/2\ s)sds = \frac{-16 + 3\pi^2}{8\pi} \cong 0.54$, we obtain

$$\int_{\mathbb{R}^2_+} R_h(t, T_1, T_8)dT_1dT_8 = \exp(-(s+r)t) \cdot \int_{C_{S_t}} R_{h0}(T_1, T_8)dT_1dT_8 =$$

$$\exp(-(s+r)t)\int_{C_{St}} cos^4\Big((\pi\sqrt{(T_{10}(T_1, t) + s_0/s)^2 + (T_{80}(T_8, t) + r_0/r)^2}\,(2p(0))^{-1}\Big)dT_1dT_8 =$$

$$\exp(-(s+r)t)\int_{C_{S0}} cos^4\Big((\pi\sqrt{(T_{10} + s_0/s)^2 + (T_{80} + r_0/r)^2}\,(2p_0)^{-1}\Big)\frac{dT_1}{dT_{10}}(t)\frac{dT_8}{dT_{80}}(t)dT_{10}dT_{80} =$$

$$\exp(-(s+r)t)\exp((s+r)t)\int_{C_{S0}} cos^4\Big((\pi\sqrt{(T_{10} + s_0/s)^2 + (T_{80} + r_0/r)^2}\,(2p_0)^{-1}\Big)dT_{10}dT_{80} =$$

$$2\pi\int_0^{p_0} cos^4\Big(\pi h(2p_0)^{-1}\Big)hdh = p_0^2\Big(2\pi\int_0^1 cos^4(\pi/2s)sds\Big) \cong 0.54p_0^2.$$

Summing up, the support of the function R_h becomes exponentially in time concentrated around the point $(-s_0/s, -r_0/r)$ and its area is equal to $\exp((s+r)t)\pi p_0^2$. On the other hand, the integral $\int_{\mathbb{R}^2_+} R_h(t, T_1, T_8) dT_1 dT_8$ is constant and equal approximately to $0.54 p_0^2$. It follows that, in the considered case

$$R_h(t, T_1, T_8) \underset{t\to\infty}{\longrightarrow} 0.54 p_0^2 \delta(-s_0/s, -r_0/r).$$

This is in agreement with the fact that, according to (8.10)

$$R_h(t, T_1, T_8) = \exp(-2st) cos^4\Big((\pi\sqrt{(T_1 + s_0/s)^2 \exp(-2st) + (T_8 + s_0/s)^2 \exp(-2st)}\,(2p(0))^{-1}\Big) \tag{8.11}$$

if $\sqrt{(T_1 + s_0/s)^2 \exp(-2st) + (T_8 + s_0/s)^2 \exp(-2st)} \le p(0)$ and 0 otherwise. Thus the maximal value of R_h grows as fast as $\exp(-2st)$. The cross sections of the graphs of the function R_h as given by the right hand side of (8.11) for $s = -2$, $s_0 > 2$ and $p(0) = 1$ for three times $t = 0$, $t = 0.5$ and $t = 1$ are shown in Fig.4. By cross sections we mean here cross sections with planes perpendicular to the (T_1, T_8)-plane passing through the point $(-s_0/s, -r_0/r)$.

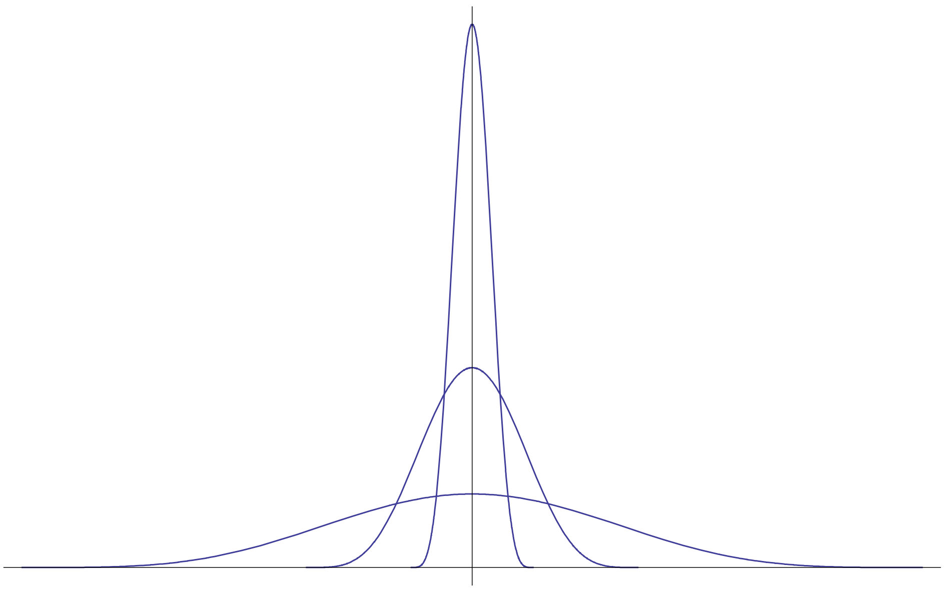

Figure 4: *Cross sections of the graphs of R_h defined by (8.11) with $s = -2$, $s_0 = 3$, $p(0) = 1$ for $t = 0$ (the flattest curve), $t = 0.5$ and $t = 1$ (the steepest curve).*

.

Now, the two remarks should be made.

Remark It should be emphasized that the factorization property concerns only solutions to the homogeneous equation. □

Remark It should be noted that for Γ and B independent of t, the result of Lemma 8.1 can be recovered via the analysis of equality (3.52). Thus, it follows from the assumption (8.4) that

$$R(t, x, T_1, T_8) = \Big(\int_{\mathbb{R}^3} G_0(t, x; 0, \xi) R_{p0}(x) d\xi\Big) \cdot \mathcal{K}(T_1, T_8; t) R_{h0}(T_{10}(T_1, t), T_{80}(T_8, t)) \tag{8.12}$$

Let us note, as before, that we can write

$$\frac{\partial R_h}{\partial t} = -\Big(\frac{\partial \Gamma(T_1)}{\partial T_1} + \frac{\partial B(T_8)}{\partial T_8}\Big) R_h - \Gamma(T_1(s)) \frac{\partial}{\partial T_1} R_h - B(T_8) \frac{\partial}{\partial T_8} R_h. \tag{8.13}$$

Let us consider the above equation on the characteristic curves $[0, T] \ni t \mapsto (t, T_1(t), T_8(t))$. Using the second equation of (21) [13, 3.2.2], we obtain for $t \in (0, T]$:

$$\frac{dR_h}{dt} = -\Big(\frac{\partial\Gamma(T_1(t))}{\partial T_1} + \frac{\partial B(T_8(t))}{\partial T_8}\Big)R_h. \tag{8.14}$$

Considering the characteristic curve starting for $t = 0$ from (T_{10}, T_{80}) and defining

$$\hat{R}_h := \exp\Big(\int_0^t \frac{\partial\Gamma(T_1(s))}{\partial T_1} ds\Big) \cdot \exp\Big(\int_0^t \frac{\partial B(T_8(t))}{\partial T_8(s)} ds\Big) \cdot R_h, \tag{8.15}$$

we obtain the equation

$$\frac{d\hat{R}_h}{dt} = 0 \tag{8.16}$$

with the initial condition $\hat{R}_h(0) = R_{h0}(T_{10}, T_{80})$. We thus obtain:

$$R_h(t, T_1(t), T_8(t)) = \exp\Big(-\int_0^t \frac{\partial\Gamma(T_1(s))}{\partial T_1} ds\Big) \cdot \exp\Big(-\int_0^t \frac{\partial B(T_8(t))}{\partial T_8(s)} ds\Big) \cdot R_{h0}(T_{10}, T_{80}).$$

Using Remark after (3.32), we have

$$\exp\Big(-\int_0^t \frac{\partial\Gamma}{\partial T_1}\Big(T_1(T_{10}, \tau)\Big)\, d\tau\Big) = \frac{\Gamma(T_{10}(T_1, t))}{\Gamma(T_1)},$$

and

$$\exp\Big(-\int_0^t \frac{\partial B}{\partial T_8}\Big(T_8(T_{80}, \tau)\Big)\, d\tau\Big) = \frac{B(T_{80}(T_1, t))}{B(T_8)}.$$

It follows that

$$R_h = \mathcal{K}(T_1, T_8, t) \cdot R_{h0}(T_{10}(T_1, t), T_{80}(T_8, t))$$

in agreement with (8.12). □

Now, referring to the second sentence of this section, we will give a very simple example of a solution to the hyperbolic counterpart of Eq.(3.1). Now, starting from (8.13), we can generalize this analysis to the case of Γ and B depending on t. In this case we have

$$\frac{\partial R_h}{\partial t} = -\Big(\frac{\partial\Gamma(T_1, t)}{\partial T_1} + \frac{\partial B(T_8, t)}{\partial T_8}\Big)R_h - \Gamma(T_1(s), t)\, \frac{\partial}{\partial T_1}\, R_h - B(T_8, t)\frac{\partial}{\partial T_8}\, R_h. \tag{8.17}$$

Not to lose conciseness, we tacitly assume that Assumption 3.2 is satisfied for all $t \in [0, T]$.

Let us consider the above equation on the characteristic curves $[0, T] \ni t \mapsto (t, T_1(t), T_8(t))$. This time they are given by the equations:

$$\begin{aligned} \frac{dT_1}{dt} &= \Gamma(T_1, t), \quad T_1(0) = T_{10}, \\ \frac{dT_8}{dt} &= B(T_8, t), \quad T_{80}(0) = T_{80}. \end{aligned} \tag{8.18}$$

Using the second equation of (21) [13, 3.2.2], we obtain for $t \in (0, T]$:

$$\frac{dR_h}{dt} = -\Big(\frac{\partial\Gamma(T_1(t), t)}{\partial T_1} + \frac{\partial B(T_8(t), t)}{\partial T_8}\Big)R_h. \tag{8.19}$$

Considering the characteristic starting for $t = 0$ from (T_{10}, T_{80}) and defining

$$\hat{R}_h := \exp\Big(\int_0^t \frac{\partial\Gamma(T_1(T_{10}, s), s)}{\partial T_1} ds\Big) \cdot \exp\Big(\int_0^t \frac{\partial B(T_8(T_{80}, s), s)}{\partial T_8} ds\Big) \cdot R_h \tag{8.20}$$

we obtain the equation

$$\frac{d\hat{R}_h}{dt} = 0 \tag{8.21}$$

with the initial condition $\hat{R}_h(0) = R_{h0}(T_{10}, T_{80})$. Consequently

$$R_h(t, T_1(t), T_8(t)) = \exp\Big(- \int_0^t \frac{\partial \Gamma(T_1(s), s)}{\partial T_1} ds \Big) \cdot \exp\Big(- \int_0^t \frac{\partial B(T_8(t), s)}{\partial T_8(s)} ds \Big) \cdot R_{h0}(T_{10}, T_{80}).$$

Below, we will consider the simple but relatively general case:

$$\Gamma(T_1, t) = p_1(t)\Gamma_*(T_1) \quad \text{and} \quad B(T_8, t) = p_8(t) B_*(T_8) \tag{8.22}$$

with $p_1(t) > 0$, $p_8(t) > 0$ for all $t \in [0, T]$.

Under this assumption, we have, by means of (8.18) and (8.22),

$$\begin{aligned} &\exp\Big(- \int_0^t \frac{\partial \Gamma(T_1(s), s)}{\partial T_1} ds \Big) = \exp\Big(- \int_0^t \frac{\partial \Gamma_*(T_1(s))}{\partial T_1} p_1(s) ds \Big) = \\ &\exp\Big(- \int_{T_{10}}^{T_1} \frac{\partial \Gamma_*(T_1(s))}{\partial T_1} dT_1 \Big) = \frac{\Gamma_*(T_{10}(T_1, t))}{\Gamma_*(T_1)} = \frac{\partial T_{10}}{\partial T_1}(t) \\ &\Big(\quad = \frac{\Gamma_*(T_{10}(T_1, t)) p_1(t)}{\Gamma_*(T_1) p_1(t)} = \frac{\Gamma(T_{10}(T_1, t), t)}{\Gamma(T_1, t)} \quad \Big), \end{aligned} \tag{8.23}$$

where we took into account that $T_1(0) = T_{10}$. Likewise,

$$\begin{aligned} &\exp\Big(- \int_0^t \frac{\partial B(T_8(s), s)}{\partial T_8} ds \Big) = \exp\Big(- \int_0^t \frac{\partial B_*(T_8(s))}{\partial T_8} p_8(s) ds \Big) = \\ &\exp\Big(- \int_{T_{80}}^{T_8} \frac{\partial B_*(T_8(s))}{\partial T_8} dT_8 \Big) = \frac{B_*(T_{80}(T_8, t))}{B_*(T_8)} = \frac{\partial T_{80}}{\partial T_8}(t) \\ &\Big(\quad = \frac{B_*(T_{80}(T_8, t)) p_1(t)}{B_*(T_8) p_8(t)} = \frac{B(T_{80}(T_8, t), t)}{B(T_8, t)} \quad \Big), \end{aligned} \tag{8.24}$$

To show that the function

$$R(t, x, T_1, T_8) = \int_{\mathbb{R}^3} G_0(t, x; 0, \xi) \cdot \frac{\Gamma_*(T_{10}(T_1, t))}{\Gamma_*(T_1)} \frac{B_*(T_{80}(T_8, t))}{B_*(T_8)} R_0(\xi, T_{10}(T_1, t), T_{80}(T_8, t))\, d\xi, \tag{8.25}$$

is a solution to Eq.(8.1), we can use the modification of the proof of Lemma 3.7. The modification consists in taking into account the fact that in the considered case, according to (8.23),(8.24), we have:

$$\frac{1}{\Gamma_*(T_1)} \cdot \frac{dT_1}{dT_{10}} \cdot \frac{\partial T_{10}(T_1, t)}{\partial t} = -p_1(t)$$

and

$$\frac{1}{B_*(T_8)} \cdot \frac{\partial T_8}{\partial T_{80}} \cdot \frac{dT_{80}(T_8, t)}{dt} = -p_8(t).$$

9 Extension to the equation with added diffusion terms

Let us consider the equation with additional diffusional terms with respect to T_1 and T_8:

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R + \varepsilon^2 \Big(\frac{\partial^2 R}{\partial T_1^2} + \frac{\partial^2 R}{\partial T_8^2} \Big) - \frac{\partial}{\partial T_1} \left(\Gamma(T_1)\, R \right) - \frac{\partial}{\partial T_8} \left(B(T_8)\, R \right) + f(t, x, T_1, T_8) \tag{9.1}$$

where $\varepsilon \geq 0$.

Remark In this section we will tacitly assume that the initial data are uniformly compactly supported with respect to (T_1, T_8), that is to say, there exist a compact set $\mathcal{S}_0 \subset \overline{\mathbb{R}^2_+}$ such that

$$\text{support}_{(T_1, T_8)} R_0(x, T_1, T_8) \subset \mathcal{S}_0$$

for all $x \in \mathbb{R}^3$. $\square$

The following lemmas generalize Lemma 3.8 and 3.9 to the case of Eq.(9.1).

Lemma 9.1. *The function*

$$R(t,x,T_1,T_8;\varepsilon) =$$
$$\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^3} G_0(t,x;0,\xi)Q_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)Q_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t)R_0(\xi,T_{10}(\mathcal{T}_1,t),T_{80}(\mathcal{T}_8,t))d\xi d\mathcal{T}_1 d\mathcal{T}_8, \tag{9.2}$$

where $\mathcal{K}(T_1,T_8;t)$ is defined by (3.53) is a solution to Eq. (9.1) with $f\equiv 0$ with the initial condition

$$R(0,x,T_1,T_8) = R_0(x,T_1,T_8)$$

and with the boundary conditions

$$R(t,x,T_1=0,T_8)=0, \quad R(t,x,T_1,T_8=0)=0$$

Here G_0 is given by (3.15), whereas $Q_{1\varepsilon}$ and $Q_{8\varepsilon}$ are Green's functions of the heat equation for the half lines $T_1 \geq 0$ and $T_8 \geq 0$, i.e. solutions to the equations

$$\frac{\partial Q_{k\varepsilon}}{\partial t} - \varepsilon^2 \frac{\partial^2 Q_{k\varepsilon}}{\partial^2 T_k} = \delta(t)\delta(T_k - \mathcal{T}_k)$$

with $T_k, \mathcal{T}_k \geq 0$ and $k = 1, 8$.

The proof follows by simple extension of the arguments used in the proof of Lemma 3.8.

Remark The explicit form of the functions $Q_{k\varepsilon}$ can be found, e.g. in [29, Section 7.1]

$$Q_{k\varepsilon}(t,T_k;\tau,\mathcal{T}_k) = (4\pi\varepsilon^2(t-\tau))^{-1/2} exp\Big(-\frac{|T_k-\mathcal{T}_k|^2}{4\varepsilon^2(t-\tau)}\Big) - (4\pi\varepsilon^2(t-\tau))^{-1/2} exp\Big(-\frac{|T_k+\mathcal{T}_k|^2}{4\varepsilon^2(t-\tau)}\Big) := Q^-_{k\varepsilon} - Q^+_{k\varepsilon}. \tag{9.3}$$

$\square$

In the similar way the extension of Lemma 3.9 can be shown.

Lemma 9.2. *The function*

$$u(t,x,T_1,T_8;\varepsilon) =$$
$$\int_0^t \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^3} G_0(t,x;\tau,\xi)\times$$
$$Q_{1\varepsilon}(t,T_1;\tau,\mathcal{T}_1)Q_{8\varepsilon}(t,T_8;\tau,\mathcal{T}_8)\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t-\tau)f(\tau,\xi,T_{10}(\mathcal{T}_1,t-\tau),T_{80}(\mathcal{T}_8,t-\tau))d\xi d\mathcal{T}_1 d\mathcal{T}_8\; d\tau \tag{9.4}$$

is a solution to Eq.(9.1) with zero initial condition.

Now, let us note that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2_+} Q_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)Q_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t)R_0(\xi,T_{10}(\mathcal{T}_1,t),T_{80}(\mathcal{T}_8,t))d\mathcal{T}_1 d\mathcal{T}_8 = \mathcal{K}(T_1,T_8;t)R_0(\xi,T_{10}(T_1,t),T_{80}(T_8,t)). \tag{9.5}$$

Remark It is worthwhile to emphasize that we do not pass to the limit $\tau \to t$ at the left hand side of (9.5). Instead, while considering the convergence of solution to its initial data, it is replaced by the equivalent limit $\varepsilon \to 0$. This passage guarantees that the product $\varepsilon^2(t-\tau) \to 0$. $\square$

From (9.5), it is seen that for every $(T_1, T_8) \in \overline{\mathbb{R}^2_+}$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} R(t,x,T_1,T_8;\varepsilon) = \\
&\int_{\mathbb{R}^3} G_0(t,x;0,\xi)\mathcal{K}(T_1,T_8;t)R_0(\xi,T_{10}(\mathcal{T}_1,t),T_{80}(\mathcal{T}_8,t))d\xi = R(t,x,T_1,T_8).
\end{aligned}
\tag{9.6}
$$

Likewise, using the fact that for every function $\mathcal{J}(t,\tau,\xi,\mathcal{T}_1,\mathcal{T}_8)$ of C^1 class with respect to its arguments, we have

$$
\lim_{\varepsilon\to 0} \int_{\mathbb{R}^2_+} Q_{1\varepsilon}(t,T_1;\tau,\mathcal{T}_1)Q_{8\varepsilon}(t,T_8;\tau,\mathcal{T}_8)\mathcal{J}(t,\tau,\xi,\mathcal{T}_1,\mathcal{T}_8)d\mathcal{T}_1 d\mathcal{T}_8 \to \mathcal{J}(t,\tau,\xi,T_1,T_8), \tag{9.7}
$$

it is seen that for every $(T_1, T_8) \in \mathbb{R}^2_+$

$$
\lim_{\varepsilon\to 0} u(t,x,T_1,T_8;\varepsilon) = u(t,x,T_1,T_8). \tag{9.8}
$$

In (9.6) and (9.8) denote the functions provided by Lemma (3.8) and (3.9) respectively.

Now, we will consider the behaviour of the derivatives of the functions R and u. First, let us present a convenient auxiliary result, which can be derived from the Gauss-Ostrogradskii theorem, but is more general. For convenience of the reader, its proof will be presented below.

Lemma 9.3. *Let $\mathcal{G}$ be a bounded region in $\mathbb{R}^m$, $m \geq 1$, whose boundary $S_{\mathcal{G}}$ is a closed, piecewise smooth surface which is positively oriented by a unit normal vector $\mathbf{n}$ directed outward from $\mathcal{G}$. If $f = f(y)$ is a scalar function with continuous partial derivatives at all points of $\overline{\mathcal{G}}$ (determined by appropriate limits as $y \to S_{\mathcal{G}}$). Then*

$$
\int_{S_{\mathcal{G}}} \mathbf{n}(s)f(s)ds = \int_{\mathcal{G}} \nabla f(y)dy.
$$

In particular, if n_j denotes the j-th component of the normal vector $\mathbf{n}$, then

$$
\int_{S_{\mathcal{G}}} n_j(s)f(s)ds = \int_{\mathcal{G}} \frac{\partial}{\partial y_j} f(y)dy.
$$

Proof Let $\mathbf{e}_j$, $j = 1,\ldots,m$, denote the unit versors of the Cartesian system in $\mathbb{R}^m$. We have:

$$
\nabla f(y) = \sum_{j=1}^{n} \mathbf{e}_j\, \nabla \cdot (f(y)\mathbf{e}_j),
$$

where ∇f denotes the gradient of a scalar function f and $\nabla \cdot \mathbf{g}$ denotes the divergence of a vector function $\mathbf{g}$. In this way, be means of the Gauss theorem,

$$
\begin{aligned}
&\int_{\mathcal{G}} \nabla f(y)dy = \sum_{j=1}^{n} \mathbf{e}_j \int_{\mathcal{G}} \nabla \cdot (f(y)\mathbf{e}_j)dy = \\
&\textstyle\sum_{j=1}^{n} \mathbf{e}_j \displaystyle\int_{S_{\mathcal{G}}} (f(y)\mathbf{e}_j)\cdot \mathbf{n}\; ds = \int_{S_{\mathcal{G}}} f(s)\sum_{j=1}^{n} \mathbf{e}_j[\mathbf{e}_j \cdot \mathbf{n}]\; ds = \int_{S_{\mathcal{G}}} \mathbf{n} f(s)ds.
\end{aligned}
$$

The lemma has been proved. □

Note, that using (9.3) we can write, for $k = 1, 8$,

$$
\frac{\partial Q_{k\varepsilon}}{\partial T_k} = -\frac{\partial Q^-_{k\varepsilon}}{\partial \mathcal{T}_k} - \frac{\partial Q^+_{k\varepsilon}}{\partial \mathcal{T}_k}
$$

hence

$$
\begin{aligned}
&\frac{\partial}{\partial T_k} R(t,x,T_1,T_8;\varepsilon) = \\
&-\int_{\mathbb{R}^3} G_0(t,x;0,\xi) \int_{\mathbb{R}^2_+} \frac{\partial}{\partial \mathcal{T}_k}\Big(Q^-_{k\varepsilon}(t,T_k;0,\mathcal{T}_k) + Q^+_{k\varepsilon}(t,T_k;0,\mathcal{T}_k)\Big) Q_{k_*\varepsilon}(t,T_{k*};0,\mathcal{T}_{k*})\times \\
&\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t) R_0(\xi, T_{10}(\mathcal{T}_1,t), T_{80}(\mathcal{T}_8,t)) d\mathcal{T}_1 d\mathcal{T}_8\, d\xi
\end{aligned} \tag{9.9}
$$

where k_* is an index complementary to k, i.e.

$$
k_* = \begin{cases} 8 & \text{if } k = 1, \\ 1 & \text{if } k = 8. \end{cases}
$$

It follows that

$$
\begin{aligned}
&\frac{\partial}{\partial T_k} R(t,x,T_1,T_8;\varepsilon) = \\
&-\int_{\mathbb{R}^3} G_0(t,x;0,\xi) \int_{\mathbb{R}^2_+} \frac{\partial}{\partial \mathcal{T}_k}\Big(\left[Q^-_{k\varepsilon}(t,T_k;0,\mathcal{T}_k) + Q^+_{k\varepsilon}(t,T_k;0,\mathcal{T}_k)\right] Q_{k_*\varepsilon}(t,T_{k*};0,\mathcal{T}_{k*})\times \\
&\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t) R_0(\xi, T_{10}(\mathcal{T}_1,t), T_{80}(\mathcal{T}_8,t))\Big) d\mathcal{T}_1 d\mathcal{T}_8\, d\xi + \\
&\int_{\mathbb{R}^3} G_0(t,x;0,\xi) \times \int_{\mathbb{R}^2_+} \left[Q^-_{k\varepsilon}(t,T_k;0,\mathcal{T}_k) + Q^+_{k\varepsilon}(t,T_k;0,\mathcal{T}_k)\right] Q_{k_*\varepsilon}(t,T_{k*};0,\mathcal{T}_{k*})\times \\
&\frac{\partial}{\partial \mathcal{T}_k}\Big(\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t) R_0(\xi, T_{10}(\mathcal{T},t), T_{80}(\mathcal{T}_8,t))\Big) d\mathcal{T}_1 d\mathcal{T}_8 d\xi
\end{aligned} \tag{9.10}
$$

Above, the formal integrals over $\mathbb{R}^2_+$, can be understood as as the limit of the integrals

$$
\int_{\mathbb{R}^2_+} (\cdot) d\mathcal{T}_1 d\mathcal{T}_8 = \lim_{r\to\infty} \int_{\mathcal{I}_C(r)} (\cdot) d\mathcal{T}_1 d\mathcal{T}_8,
$$

over the region $\mathcal{I}_C$ comprised within the contours C composed of the lines $\{\mathcal{T}_1 = 0,\, 0 \le \mathcal{T}_8 \le r\}$, $\{\mathcal{T}_8 = 0,\, 0 \le \mathcal{T}_1 \le r\}$, and the quarter-circle $\{\mathcal{T}_1 \ge 0,\, \mathcal{T}_8 \ge 0, \mathcal{T}_1^2 + \mathcal{T}_8^2 = r^2\}$. Let us note that due to the fact that the initial support R_0 of the function R with respect to (T_1, T_8) is compact independently of $\xi \in \overline{\Omega}$ thus, for given $t > 0$, there exists $r(t) > 1$ sufficiently large such that $R_0(\xi, T_{10}(\mathcal{T}_1,t), T_{10}(\mathcal{T}_1,t)) \equiv 0$ if $\mathcal{T}_1^2 + \mathcal{T}_8^2 \ge r(t)$. It follows that for fixed $t > 0$ and $\xi \in \mathbb{R}^3$:

$$
\left[Q^-_{k\varepsilon}(t,T_k;0,\mathcal{T}_k) + Q^+_{k\varepsilon}(t,T_k;0,\mathcal{T}_k)\right] Q_{k_*\varepsilon}(t,T_{k*};0,\mathcal{T}_{k*}) \mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t) R_0(\xi, T_{10}(\mathcal{T}_1,t), T_{80}(\mathcal{T}_8,t)) \equiv 0
$$

together with its derivatives for $\mathcal{T}_1^2 + \mathcal{T}_8^2 \ge r(t)$. Next, the function $R_0(\xi, T_{10}(\mathcal{T}_1,t), T_{80}(\mathcal{T}_8,t))$ vanish for $\mathcal{T}_1 = 0$ or $\mathcal{T}_8 = 0$, the last expression is equal to zero also on the axes $T_1 = 0$ and $T_8 = 0$. Thus, using the second identity in Lemma 9.3 with $m = 2$ and $n_j = n_k$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2_+} \frac{\partial}{\partial \mathcal{T}_k} \\
&\Big(\left[Q^-_{k\varepsilon}(t,T_k;0,\mathcal{T}_k) + Q^+_{k\varepsilon}(t,T_k;0,\mathcal{T}_k)\right] Q_{k_*\varepsilon}(t,T_{k*};0,\mathcal{T}_{k*}) \mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t) R_0(\xi, T_{10}(\mathcal{T}_1,t), T_{80}(\mathcal{T}_8,t))\Big) d\mathcal{T}_1 d\mathcal{T}_8 \\
&= \lim_{r\to\infty} \int_{\mathcal{I}_C(r)} \frac{\partial}{\partial \mathcal{T}_k} \\
&\Big(\left[Q^-_{k\varepsilon}(t,T_k;0,\mathcal{T}_k) + Q^+_{k\varepsilon}(t,T_k;0,\mathcal{T}_k)\right] Q_{k_*\varepsilon}(t,T_{k*};0,\mathcal{T}_{k*}) \mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t) R_0(\xi, T_{10}(\mathcal{T}_1,t), T_{80}(\mathcal{T}_8,t))\Big) d\mathcal{T}_1 d\mathcal{T}_8 \\
&= \int_{C(r)} \left[Q^-_{k\varepsilon}(t,T_k;0,\mathcal{T}_k) + Q^+_{k\varepsilon}(t,T_k;0,\mathcal{T}_k)\right] \\
&\qquad \times Q_{k_*\varepsilon}(t,T_{k*};0,\mathcal{T}_{k*}) \mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t) R_0(\xi, T_{10}(\mathcal{T}_1,t), T_{80}(\mathcal{T}_8,t)) n_k ds(r),
\end{aligned}
$$

where $ds(r)$ is the infinitesimal arc length over the circle $C(0,r)$. It follows from (9.10) that

$$\frac{\partial}{\partial T_k} R(t,x,T_1,T_8;\varepsilon) = \int_{\mathbb{R}^3} G_0(t,x;0,\xi) \int_{\mathbb{R}^2_+} \left[Q^-_{k\varepsilon}(t,T_1;0,\mathcal{T}_1) + Q^+_{k\varepsilon}(t,T_1;0,\mathcal{T}_1)\right] Q_{k_*\varepsilon}(t,T_8;0,\mathcal{T}_8)\times$$
$$\frac{\partial}{\partial \mathcal{T}_k}\Big(\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t)R_0(\xi,T_{10}(\mathcal{T}_1,t),T_{80}(\mathcal{T}_8,t))\Big)d\mathcal{T}_1 d\mathcal{T}_8\, d\xi \tag{9.11}$$

Likewise, we can show that

$$\frac{\partial}{\partial T_k} u(t,x,T_1,T_8;\varepsilon) = \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^2_+} G_0(t,x;\tau,\xi)\ \left[Q^-_{k\varepsilon}(t,T_k;0,\mathcal{T}_k) + Q^+_{k\varepsilon}(t,T_k;0,\mathcal{T}_k)\right] Q_{k_*\varepsilon}(t,T_{k*};0,\mathcal{T}_{k*})$$
$$\times\frac{\partial}{\partial \mathcal{T}_k}\Big(\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t-\tau)f(\tau,\xi,T_{10}(\mathcal{T}_1,t-\tau),T_{80}(\mathcal{T}_8,t-\tau))d\mathcal{T}_1 d\mathcal{T}_8\Big)\ d\xi\ d\tau$$

Finally, using the identities (9.6) and (9.8) one can prove that

$$\lim_{\varepsilon\to 0} \frac{\partial}{\partial T_k} R(t,x,T_1,T_8;\varepsilon) = \frac{\partial}{\partial T_k} R(t,x,T_1,T_8) \tag{9.12}$$

and

$$\lim_{\varepsilon\to 0} \frac{\partial}{\partial T_k} u(t,x,T_1,T_8;\varepsilon) = \frac{\partial}{\partial T_k} u(t,x,T_1,T_8). \tag{9.13}$$

This follows from the form of the functions $Q_{1\varepsilon}$, $Q_{8\varepsilon}$ given by (9.3), the Remark after (9.5), and point 2 of Lemma 3.4, which result in the following simple lemma.

Lemma 9.4. *Suppose that the support of the function $g(T_1,\mathcal{T}_8) \in C^0(\mathbb{R}^2)$ is compact and contained in $\mathbb{R}^2_+$. Then*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \times$$
$$\Big(G^-_{1\varepsilon}(t,T_1;0,\mathcal{T}_1) \pm G^+_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\Big)\Big(G^-_{8\varepsilon}(t,T_8;0,\mathcal{T}_8) \pm G^+_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\Big)g(\mathcal{T}_1,\mathcal{T}_8)d\mathcal{T}_1 d\mathcal{T}_8 =$$
$$= g(T_1,T_8).$$

Proof Due to the compactness of the support of the function g we can write

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \times$$
$$\Big(G^-_{1\varepsilon}(t,T_1;0,\mathcal{T}_1) \pm G^+_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\Big)\Big(G^-_{8\varepsilon}(t,T_8;0,\mathcal{T}_8) \pm G^+_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\Big)g(\mathcal{T}_1,\mathcal{T}_8)d\mathcal{T}_1 d\mathcal{T}_8 =$$
$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+} \Big(G^-_{8\varepsilon}(t,T_8;0,\mathcal{T}_8) \pm G^+_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\Big)\times$$
$$\Big[\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+} \Big(G^-_{1\varepsilon}(t,T_1;0,\mathcal{T}_1) \pm G^+_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\Big)g(\mathcal{T}_1,\mathcal{T}_8)d\mathcal{T}_1\Big]d\mathcal{T}_8$$

As $g(T_1,\mathcal{T}_8) \equiv 0$ for $T_1 \le 0$ and all $\mathcal{T}_8 \in \mathbb{R}$, we have $H(-T_1)g(T_1,\mathcal{T}_8) = H(T_1)g(-T_1,\mathcal{T}_8) \equiv 0$ hence by point 2 of Lemma 3.4

$$\lim_{t\to 0}\int_{\mathbb{R}_+}\Big(G^-_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\pm G^+_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\Big)g(\mathcal{T}_1,\mathcal{T}_8)d\mathcal{T}_1 =$$
$$\lim_{t\to 0}\int_{\mathbb{R}}\Big(G^-_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\pm G^+_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\Big)H(\mathcal{T}_1)g(\mathcal{T}_1,\mathcal{T}_8)d\mathcal{T}_1 =$$
$$H(T_1)g(T_1,\mathcal{T}_8)\pm H(T_1)g(-T_1,\mathcal{T}_8)=g(T_1,\mathcal{T}_8).$$

Similarly, as $g(T_1,\mathcal{T}_8)\equiv 0$ for $\mathcal{T}_8\le 0$ and all $T_1\in\mathbb{R}$, we have $H(-\mathcal{T}_8)g(T_1,\mathcal{T}_8)=H(\mathcal{T}_8)g(T_1,-\mathcal{T}_8)\equiv 0$ hence

$$\lim_{t\to 0}\int_{\mathbb{R}_+}\Big(G^-_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\pm G^+_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\Big)g(T_1,\mathcal{T}_8)d\mathcal{T}_8 =$$
$$\lim_{t\to 0}\int_{\mathbb{R}}\Big(G^-_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\pm G^+_{8\varepsilon}(t,T_8;0,\mathcal{T}_8)\Big)H(\mathcal{T}_8)g(T_1,\mathcal{T}_8)d\mathcal{T}_8 =$$
$$H(T_8)g(T_1,T_8)\pm H(T_8)g(T_1,-T_8)=g(T_1,T_8).$$

The lemma is proved. □

The same analysis can be carried out in case of the second derivatives with respect to T_k, $k=1,8$. Thus repeating twice the arguments presented above we obtain:

$$\begin{aligned}&\frac{\partial^2}{\partial T_1\partial T_8}R(t,x,T_1,T_8;\varepsilon)=\\&\int_{\mathbb{R}^3}G_0(t,x;0,\xi)\int_{\mathbb{R}^2_+}\left[Q^-_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)+Q^+_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\right]\left[Q^-_{8\varepsilon}(t,T_1;0,\mathcal{T}_1)+Q^+_{8\varepsilon}(t,T_1;0,\mathcal{T}_1)\right]\times\\&\frac{\partial^2}{\partial\mathcal{T}_1\partial\mathcal{T}_8}\Big(\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t)R_0(\xi,T_{10}(\mathcal{T}_1,t),T_{80}(\mathcal{T}_8,t))\Big)d\mathcal{T}_1d\mathcal{T}_8\,d\xi.\end{aligned}\tag{9.14}$$

Next, due to the fact that, for $k=1,8$,

$$\frac{\partial^2 Q_{k\varepsilon}}{\partial T_k^2}=\frac{\partial^2 Q_{k\varepsilon}}{\partial\mathcal{T}_k^2},$$

we obtain

$$\begin{aligned}&\frac{\partial^2}{\partial T_k^2}R(t,x,T_1,T_8;\varepsilon)=\int_{\mathbb{R}^3}G_0(t,x;0,\xi)\int_{\mathbb{R}^2_+}Q_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\cdot Q_{8\varepsilon}(t,T_1;0,\mathcal{T}_1)\times\\&\frac{\partial^2}{\partial\mathcal{T}_k^2}\Big(\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t)R_0(\xi,T_{10}(\mathcal{T}_1,t),T_{80}(\mathcal{T}_8,t))\Big)d\mathcal{T}_1d\mathcal{T}_8\,d\xi.\end{aligned}\tag{9.15}$$

Likewise

$$\begin{aligned}&\frac{\partial^2}{\partial T_1\partial T_8}u(t,x,T_1,T_8;\varepsilon)=\\&\int_{\mathbb{R}^3}G_0(t,x;0,\xi)\int_{\mathbb{R}^2_+}\left[Q^-_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)+Q^+_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\right]\left[Q^-_{8\varepsilon}(t,T_1;0,\mathcal{T}_1)+Q^+_{8\varepsilon}(t,T_1;0,\mathcal{T}_1)\right]\times\\&\frac{\partial^2}{\partial\mathcal{T}_1\partial\mathcal{T}_8}\Big(\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t-\tau)f(\tau,\xi,T_{10}(\mathcal{T}_1,t-\tau),T_{80}(\mathcal{T}_8,t-\tau))d\mathcal{T}_1d\mathcal{T}_8\Big)d\mathcal{T}_1d\mathcal{T}_8\,d\xi.\end{aligned}\tag{9.16}$$

and

$$\frac{\partial^2}{\partial T_k^2}u(t,x,T_1,T_8;\varepsilon) = \int_{\mathbb{R}^3} G_0(t,x;0,\xi)\int_{\mathbb{R}^2_+} Q_{1\varepsilon}(t,T_1;0,\mathcal{T}_1)\cdot Q_{8\varepsilon}(t,T_1;0,\mathcal{T}_1)\times$$
$$\frac{\partial^2}{\partial \mathcal{T}_k^2}\Big(\mathcal{K}(\mathcal{T}_1,\mathcal{T}_8;t-\tau)f(\tau,\xi,T_{10}(\mathcal{T}_1,t-\tau),T_{80}(\mathcal{T}_8,t-\tau))d\mathcal{T}_1 d\mathcal{T}_8\Big)d\mathcal{T}_1 d\mathcal{T}_8\, d\xi. \tag{9.17}$$

Similarly to (9.12) and (9.13) we have, for $k,l=1,8$,

$$\lim_{\varepsilon\to 0}\frac{\partial^2}{\partial T_k\partial T_l}R(t,x,T_1,T_8;\varepsilon) = \frac{\partial^2}{\partial T_k\partial T_l}R(t,x,T_1,T_8) \tag{9.18}$$

and

$$\lim_{\varepsilon\to 0}\frac{\partial^2}{\partial T_k\partial T_l}u(t,x,T_1,T_8;\varepsilon) = \frac{\partial^2}{\partial T_k\partial T_l}u(t,x,T_1,T_8). \tag{9.19}$$

Consequently, the following lemma holds.

Lemma 9.5. *Let $R(t,x,T_1,T_8;\varepsilon)$ and $u(t,x,T_1,T_8;\varepsilon)$ denote the functions defined in (9.2) and (9.4), whereas $R(t,x,T_1,T_8)$ and $u(t,x,T_1,T_8)$ the corresponding functions defined by 3.8 and 3.9. Then, for every $t\in[0,T)$,*

$$R(t,x,T_1,T_8;\varepsilon)\to R(t,x,T_1,T_8) \quad \text{and} \quad u(t,x,T_1,T_8;\varepsilon)\to u(t,x,T_1,T_8)$$

as $\varepsilon\to 0$, in the $C^2(\overline{\Omega}\times\overline{\mathbb{R}^2_+})$ norm.

Remark It follows straightforwardly from Eq.(9.1) that also the time derivatives of the functions $R(t,x,T_1,T_8;\varepsilon)$ and $u(t,x,T_1,T_8;\varepsilon)$ tend the time derivatives of $R(t,x,T_1,T_8)$ and $u(t,x,T_1,T_8)$ as $\varepsilon\to 0$ for each $(x,T_1,T_8)\in\overline{\Omega}\times\overline{\mathbb{R}^2_+}$. □

10 Remarks on the existence of the Green's function for the Neumann problems in bounded regions

The explicit form of the Green's function with homogeneous boundary conditions of Robin type has been found for many specific bounded regions, like an interval, a sphere [26], or a rectangle [14]. However, in many standard books in the theory of parabolic differential equations, the existence of Green's function for Neumann problems is not stated. (In [15] such an existence is only mentioned as a result of [15, Problem 5, chapter 5]. Instead, in [15] or [29] (which in the context of Green's function approach is based on the results in [15]) another form of integral representation of the solution to the Neumann (second type) initial boundary value problem is presented (see (3.5) in [15, section 3, chapter 5]).

Let $\Omega\subset\mathbb{R}^m$ be a bounded domain with the boundary of class $C^{2+\upsilon}$, $\upsilon\in(0,1)$. For $\tau\geq 0$, $T>\tau$, let us consider the linear parabolic equation:

$$\begin{aligned} &Au = f(t,x) && \text{in } (\tau,T)\times\Omega,\\ &u(\tau,x)=\psi(x) && \text{on } \overline{\Omega},\\ &\frac{\partial u(t,x)}{\partial\nu(t,x)} = 0 && \text{on } (\tau,T)\times\partial\Omega, \end{aligned} \tag{10.1}$$

where ψ and f given,

$$A := L - \frac{\partial u}{\partial t}, \tag{10.2}$$

and

$$L := \sum_{i,j=1}^{m} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j(t,x)\frac{\partial}{\partial x_j} + c(t,x) \tag{10.3}$$

is a second order uniformly elliptic operator with sufficiently smooth coefficients depending on (t,x). $\nu(t,x)$ denotes the vector with components $\nu_i(t,x) = \sum a_{ij}(t,x)n_j(x)$, where $n(x) = (n_1(x),\ldots,n_m(x))$ is the unit outward vector normal to $\partial\Omega$ at x. Thus

$$\frac{\partial u(t,x)}{\partial \nu(t,x)} = \nu(t,x)\cdot\nabla u(t,x) = \sum a_{ij}(t,x)n_j(x)\frac{\partial u}{\partial x_j}(t,x). \tag{10.4}$$

In fact, the left hand side of (10.4) denotes the diffusional flux through the boundary at point $x \in \partial\Omega$. Let $\Gamma(t,x;\tau,\xi)$ denotes the fundamental solution to the first equation of system (10.1). We thus assume that Γ satisfies the equation $Au = 0$ as a function of (t,x) in $\Omega\times(0,T)$ for all $(\tau,\xi) \in \Omega\times(0,T)\cap\{\tau < t\}$ and that for any $\psi \in C^0(\overline{\Omega})$ and any $x \in \Omega$, we have

$$\lim_{t \searrow \tau} \int_\Omega \Gamma(t,x;\tau,\xi)\psi(\xi)d\xi = \psi(x). \tag{10.5}$$

In particular, if L is equal to $d_R\Delta$, then Γ is given by the right hand side of (3.15), i.e.

$$\Gamma(t,x;\tau,\xi) = \frac{1}{(4\pi d_R(t-\tau))^{3/2}} e^{-\frac{|x-\xi|^2}{4d_R(t-\tau)}}.$$

Basing on the representation of solution to system (10.1), given by (3.5) in [15, section 3, chapter 5], it is proved in [6] the existence of the Green's function for problem (10.1). Moreover, in a sense, the Green's function is defined explicitly. Thus, let

$$M_1(t,x;\tau,\xi) = -2\frac{\partial\Gamma(t,x;\tau,\xi)}{\partial\nu(t,x)}$$

and

$$M_{\rho+1}(t,x;\tau,\xi) = \int_\tau^t \int_{\partial\Omega} M_1(t,x;\sigma,\eta)M_\rho(\sigma,\eta;\tau,\xi)d\eta d\sigma.$$

Having shown the convergence of the series

$$\sum_{\rho=1}^{s_\rho} M_\rho(t,x;\tau,\xi)$$

as $s_\rho \to \infty$, it is proved in [6, Section 3] that the Green's function for (10.1) can be constructed, in a way, explicitly. Thus, if

$$\mathcal{N}(t,x;\tau,\xi) = -2\frac{\partial}{\partial\nu(t,x)}\Gamma(t,x;\tau,\xi) - 2\sum_{\rho=1}^{\infty}\int_\tau^t\int_{\partial\Omega} M_\rho(t,x;\sigma,\eta)\frac{\partial}{\partial\nu(\sigma,\eta)}\Gamma(\sigma,\eta;\tau,\xi)d\eta d\sigma, \tag{10.6}$$

then the following lemma holds.

Lemma 10.1. *The Green's function for problem (10.1) is equal to*

$$G(t,x;\tau,\xi) = \int_\tau^t\int_{\partial\Omega} \Gamma(t,x;\sigma,\eta)\mathcal{N}(\sigma,\eta;\tau,\xi)d\eta d\sigma + \Gamma(t,x;\tau,\xi). \tag{10.7}$$

This function satisfies the identity

$$G(t,x;\tau,\xi) = \int_\Omega G(t,x;\sigma,\eta)G(\sigma,\eta;\tau,\xi)d\eta \quad \tau < \sigma < t$$

and if $c(t,x) = 0$, then

$$\int_\Omega G(t,x;\tau,\xi)d\xi = 1.$$

The solution to problem (10.1) can be represented in the form:

$$u(t,x) = \int_\Omega G(t,x;\tau,\xi)\psi(\xi)d\xi + \int_\tau^t \left(\int_\Omega G(t,x;\xi,s)f(x,s)d\xi \right) ds.$$

Remark It follows from Lemma 10.1 and (10.5) that the Green's function for the bounded region Ω (with sufficiently smooth boundary) satisfies the properties corresponding to points 1,2,3,4 of Lemma 3.4. □

Remark Suppose that for all $\xi \in \Omega$, $t > \tau$, the fundamental solution $\Gamma(t,x;\tau,\xi)$ of the equation $Au = 0$ (which is, in general, different than the heat kernel given by (3.18)) satisfies the boundary conditions

$$\frac{\partial \Gamma(t,x;\tau,\xi)}{\partial \nu(t,x)} = 0 \qquad\qquad \text{on } (0,T) \times \partial\Omega.$$

Then, according to (10.6) and (10.7), $G(t,x;\tau,\xi) = \Gamma(t,x;\tau,\xi)$. □

Remark Let us note that, in the case $a_{ij} = \delta_{ij} d_R$, then, due to definition (10.4)), the condition $\dfrac{\partial u(t,x)}{\partial \nu(t,x)} = 0$ implies the homogeneous Neumann boundary condition

$$\frac{\partial\, u}{\partial n} = 0 \quad \text{for } n = n(x),\ x \in \partial\Omega.$$

□

11 The case of bounded regions

In view of section 10, we can generalize our previous results to bounded regions. To be more precise, the following statement holds.

Theorem 11.1. *Let $\Omega \subset \mathbb{R}^j$, $1 \le j \le 3$ be a bounded domain with sufficiently smooth boundary. Then Lemma 3.8, Lemma 3.9, uniqueness results in section 4, and the convergence statements in section 9 remain valid.*

Proof The proof of this statement is due to the fact that all the expressions exploited in above can be used modulo the formal change of $G_0 \mapsto G$, where G is the Green's function for the problem (10.1) for $L = d_R \nabla^2$. To prove it suffices to apply the previous analysis to the integrals over finite space regions Ω. □

In particular, the unique solution to the initial value problem for the homogeneous and inhomogeneous equation corresponding to (3.1) is given in the following lemma.

Theorem 11.2. *Let $m = 1,2,3$. Let Ω be a bounded region with the boundary $\partial\Omega$ in $C^{2+\upsilon}$ class. Let G denotes the Green's function for the problem (10.1) with $L = d_R\nabla^2$. Suppose that Assumption 3.3 holds for all $x \in \overline{\Omega}$, and that for all $(T_1, T_8) \in \overline{\mathbb{R}^2_+}$, the function $R_0 : \overline{\Omega} \times \overline{\mathbb{R}^2_+}$, satisfies*

$$\frac{\partial}{\partial n(x)} R_0(x, T_1, T_8) = 0 \quad \textit{for } x \in \partial\Omega.$$

Then, the function

$$R(t,x,T_1,T_8) = \int_\Omega G(t,x;0,\xi)\mathcal{K}(T_1,T_8;t)R_0(\xi, T_{10}(T_1,t), T_{80}(T_8,t))d\xi\,, \tag{11.1}$$

where

$$\mathcal{K}(T_1,T_8;t) := \frac{\Gamma(T_{10}(T_1,t))}{\Gamma(T_1)} \frac{B(T_{80}(T_8,t))}{B(T_8)}, \tag{11.2}$$

is a solution to the equation

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1}\left(\Gamma(T_1;x)\, R\right) - \frac{\partial}{\partial T_8}\left(B(T_8;x)\, R\right) \tag{11.3}$$

with the initial condition

$$R(0,x,T_1,T_8) = R_0(x,T_1,T_8)$$

and the homogeneous Neumann boundary conditions

$$\frac{\partial}{\partial n(x)} R(t,x,T_1,T_8) = 0 \quad \text{for } x \in \partial\Omega.$$

Next, the function

$$u(t,x,T_1,T_8) = \int_0^t \Big(\int_\Omega \mathcal{K}(T_1,T_8;t-\tau) G(t,x;\tau,\xi) f(\tau,\xi,T_{10}(T_1,t-\tau),T_{80}(T_8,t-\tau)) d\xi \Big) d\tau \quad (11.4)$$

is the solution to the non-homogeneous equation

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1} \left(\Gamma(T_1)\, R\right) - \frac{\partial}{\partial T_8} \left(B(T_8)\, R\right) + f(t,x,T_1,T_8). \quad (11.5)$$

with zero initial conditions and the homogeneous Neumann boundary conditions. If $R_0 \in C^{\upsilon,2,2}_{x,T_1,T_8}$ *and* $f \in C^{\upsilon/2,\upsilon,2,2}_{t,x,T_1,T_8}$*, whereas* Γ *and* B *are of* C^3 *class of their arguments (in the corresponding domains), then the functions given by the right hand sides of (11.1) and (11.4) are bounded in the norm of the space* $C^{1+\upsilon/2,2+\upsilon,2}_{t,x,(T_1,T_8)}([0,T]\times\overline{\Omega}\times\overline{\mathbb{R}^2_+})$*. These solutions are unique in the space of functions* $\mathcal{W}_{2B}$ *which is defined similarly to the space* $\mathcal{W}_2$ *before Lemma 4.2 by replacing* $\mathbb{R}^3$ *with* $\overline{\Omega}$*.*

12 The case of Γ and B depending on x and t

In this section we will consider the case of the functions Γ and B depending on x and t. For technical reasons, in view of section 11, we will assume that $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$. Moreover, supported by the fact that the number of receptors on the cell membrane is bounded, we will compactify the region in the (T_1,T_8) space the domain $\mathcal{D}^{rec}$ of Γ and B as

$$\mathcal{D}^{rec} = \Big\{(T_1,T_8) : T_1 \in [0,T_{1max}], T_8 \in [0,T_{8max}\Big\},$$

and take:

$$\Gamma, B : \mathcal{D}^{rec} \times [0,T] \times \overline{\Omega} \mapsto \mathbb{R}.$$

We will thus consider the equation

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1} \left(\Gamma(T_1;x;t)\, R\right) - \frac{\partial}{\partial T_8} \left(B(T_8;x;t)\, R\right) \quad (12.1)$$

with Γ and B depending **also** on x and t. As before, this equation is supplemented with the initial data R_0, together with homogeneous Neumann boundary conditions with respect to $\partial\Omega$ and homogeneous Dirichlet boundary condition on the boundary of the set $\mathbb{R}^2_+$.

Assumption 12.1. *R satisfies the following boundary conditions:*

$$(\nabla R)\cdot n(x) = 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad R(x,T_1,T_8) = 0 \quad \text{for} \quad (T_1,T_8) \in \partial\mathcal{D}^{rec}. \quad (12.2)$$

Let $\alpha \in (1/2,1)$ be fixed.

Assumption 12.2. $R_0 \in C^{2+\alpha,2,2}_{x,T_1,T_8}$ *satisfies conditions (12.2).*

Let $\theta > 0$ be fixed, but at our disposal. Let us also choose a point $x_0 \in \overline{\Omega}$ such that there exists a pair $(T_1,T_8) \in \mathcal{D}^{rec}$ for which

$$R_0(x_0,T_1,T_8) > 0.$$

To prove a local in time existence of solutions to Eq.(12.1), let us define a mapping:

$$\tilde{R} \mapsto \mathcal{L}(\tilde{R}) = \mathcal{R}(R_0,\tilde{R}), \quad (12.3)$$

where $\mathcal{R}$ is a solution to the equation

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1}\left(\Gamma(T_1; x_0; 0)\, R\right) - \frac{\partial}{\partial T_8}\left(B(T_8; x_0; 0)\, R\right) + \mathcal{F}(t, x, T_1, T_8, \tilde{R}), \tag{12.4}$$

where

$$\mathcal{F}(t, x, T_1, T_8, \tilde{R}) :=$$

$$-\tfrac{\partial}{\partial T_1}\Big(\Gamma(T_1; x; t)\, \tilde{R}\Big) - \tfrac{\partial}{\partial T_8}\Big(B(T_8; x; t)\, \tilde{R}\Big) + \tfrac{\partial}{\partial T_1}\Big(\Gamma(T_1; x_0; 0)\, \tilde{R}\Big) + \tfrac{\partial}{\partial T_8}\Big(B(T_8; x_0; 0)\, \tilde{R}\Big)$$

Let R_* denote the solution defined on $[0, T]$ to Eq. (12.4) with the term $\mathcal{F} \equiv 0$.

It is seen that the fixed points of the mapping (12.3) are equivalent to solutions of Eq.(12.1) with the above determined initial and boundary conditions.

Definition 12.3. *Let* $J = \dfrac{1}{2}\|R_0 - R_*\|$, *where* $\|\cdot\|$, *where denotes the norm in the space*

$$C^{1+\alpha, 2+\alpha, 2, 2}_{t, x, T_1, T_8}([0, T] \times \overline{\Omega} \times \mathcal{R}^{rec}). \tag{12.5}$$

Definition 12.4. *For* $\theta \in (0, T]$ *and* $\gamma = \alpha/2$, *let*

$$\mathcal{H}^1_\theta := \Big\{ h \in C^{1+\gamma, 2+\gamma, 1, 1}_{t, x, T_1, T_8}([0, \tau] \times \overline{\Omega} \times \mathcal{D}^{rec}) : h \textit{ satisfies conditions (12.2)} \Big\}.$$

Definition 12.5.

$$\mathcal{H}^2_{\theta J} := \Big\{ v \in \mathcal{H}^1_\theta,\ v \in C^{1+\gamma, 2+\gamma, 2, 2}_{t, x, T_1, T_8},\ \|v - R_0\|_2 \le J \Big\},$$

where $\|\cdot\|_2$ *denotes the norm in the space*

$$C^{1+\gamma, 2+\gamma, 2, 2}_{t, x, T_1, T_8}([0, T] \times \overline{\Omega} \times \mathcal{R}^{rec}).$$

For convenience, let us quote the version the Schauder fixed point theorem, which will be used below.

Theorem 12.6. *([13, 9.2.2,Theorem 3], see also [33]) Let* X *be a Banach space. Suppose that* $K \subset X$ *is compact and convex, and assume also that* $\mathcal{L} : K \mapsto K$ *is continuous. Then* $\mathcal{L}$ *has a fixed point in* K.

In our analysis, we will take $\mathcal{H}^1_\theta$ as X and $\mathcal{H}^2_{\theta J}$ as K.

First, we note that the mapping $\mathcal{L}$ is continuous. Next, the set $\mathcal{H}^2_{\theta J}$ is compact and convex. To proceed, we must show that there exists $\theta > 0$ sufficiently small. This follows from Theorem 11.2, with α replaced by $\gamma = \alpha/2$ (which is justified as $\gamma < \alpha$) together with the Schauder estimates applied to the parabolic part of Eq.(12.4) and the properties of solutions to system determining the bicharacteristics of the hyperbolic part.

First, from the explicit form of the solution to the inhomogeneous problem with $f(t, x, T_1, T_8)$ replaced by $\mathcal{F}$, we conclude that for $\tilde{R} \in \mathcal{H}^2_{\theta J}$, $\Gamma \in C^{3,1,1}_{T_1, x, t}$ and $B \in C^{3,1,1}_{T_8, x, t}$, $\mathcal{R}(R_0, \tilde{R})$ is of class C^2 with respect to T_1, T_8. Moreover, given $\tilde{R}$, the function $\mathcal{F}$ is of $C^{1,1}_{x,t}$ class for each (T_1, T_8). Let us note that Eq.(12.4) can be written as

$$\begin{aligned} \frac{\partial R}{\partial t} &= d_R \nabla^2 R - \Big(\tfrac{\partial}{\partial T_1}\Gamma(T_1; x_0; 0)\Big) R - \Big(\tfrac{\partial}{\partial T_8} B(T_8; x_0; 0)\Big) R + \\ &\Big(- \Gamma(T_1; x_0; 0) \Big(\tfrac{\partial}{\partial T_1} R\Big) - B(T_8; x_0; 0) \Big(\tfrac{\partial}{\partial T_8} R\Big) + \mathcal{F}(t, x, T_1, T_8, \tilde{R}(t, x, T_1, T_8))\Big), \end{aligned} \tag{12.6}$$

and its solution is given by the sum of (11.1) and (11.4), where $f(t, x, T_1, T_8)$ is identified with

$$f(t, x, T_1, T_8) = \mathcal{F}(t, x, T_1, T_8, \tilde{R}(t, x, T_1, T_8)).$$

Next, from the explicit form of the function $\mathcal{R}(R_0, \tilde{R})$, together with Lemmata 3.5 and 3.6, we see that its first and second derivatives of with respect to T_1 and T_8 are uniformly bounded on $[0,\theta]\times\overline{\Omega}\times\mathcal{R}^{rec}$. Differentiating Eq.(12.6) with respect to T_1, we obtain the equation for R_{T_1}:

$$\frac{\partial R_{T_1}}{\partial t} = d_R\nabla^2 R_{T_1} - 2\frac{\partial}{\partial T_1}\Gamma(T_1;x_0;0)R_{T_1} - \frac{\partial}{\partial T_8}B(T_8;x_0;0)R_{T_1} +$$
$$\Big\{ -\Gamma(T_1;x_0;0)R_{T_1T_1} - \Gamma(T_1;x_0;0)_{T_1T_1}R - \Gamma(T_1;x_0;0)R_{T_1} - B(T_8;x_0;0)R_{T_8T_1} + f(t,x,T_1,T_8)_{T_1T_1}\Big\}, \tag{12.7}$$

where the expression in the curly brackets is uniformly bounded in L^∞ norm. Thus using the estimate (16.48) Theorem 6.49 of section VI in [25], we conclude that for each $(T_1,T_8)\in\mathcal{R}^{rec}$, the mild solution to Eq. (12.7) satisfies the estimate

$$\|R_{T_1}\|_{C^{(1+\beta)/2,1+\beta}_{t,x}([0,\tau]\times\overline{\Omega})} \tag{12.8}$$

for all $\beta\in(0,1)$. Likewise, we obtain the estimate

$$\|R_{T_8}\|_{C^{(1+\beta)/2,1+\beta}_{t,x}([0,\tau]\times\overline{\Omega})}. \tag{12.9}$$

Recall Theorem 5.3 of Section IV in [23].

Lemma 12.7. *Let Ω be bounded open subset of $\mathbb{R}^{m_\Omega}$, $m_\Omega > 0$, and*

$$\mathcal{L} = \sum_{i,j=1}^{m_\Omega} a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{j=1}^{m_\Omega} b_j(t,x)\frac{\partial}{\partial x_j} - c(t,x).$$

Let $T>0$, $m>0$ be a non-integer number, $\partial\Omega\in C^{m+2}$, and the coefficients of the operator $\mathcal{L}$ belong to the class $C^{m/2,m}(\Omega)$. Then, for any $f\in C^{m/2,m}(\Omega)$, and $\phi = U(0,x)$ and $\Phi\in C^{m+1}(\partial\Omega)$, which satisfy compatibility conditions of order $\left[\frac{m+1}{2}\right]$, the problem

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f(t,x) \quad \text{in } (0,T)\times\Omega$$

$$u(0,x) = \phi(x) \quad \text{in } \Omega$$

$$n(x)\cdot\nabla u = \Phi(x) \quad \text{on } \partial\Omega$$

has a unique solution from $C^{1+m/2,2+m}(\Omega)$ satisfying the estimate

$$\|u\|_{C^{1+m/2,2+m}} \le c\left(\|f\|_{C^{m/2,m}((0,T)\times\Omega)} + \|\phi\|_{C^{m+2}(\Omega)} + \|\Phi\|_{C^{m+1}(\partial\Omega)}\right),$$

where, for given $T>0$, the constant c can be taken as independent of $t\in(0,T)$.

Noting that the expression in the bracket at the right hand side of Eq. (12.6) belongs at least to the class $C^{1,1}_{t,x}$, applying Lemma 12.7, and estimates (12.8) and (12.9) to Eq. (12.6), we conclude that

$$\|R\|_2 < C,$$

where, as in Definition 12.5, $\|\cdot\|_2$ denotes the norm in the space $C^{1+\alpha,2+\alpha,2,2}_{t,x,T_1,T_8}([0,T]\times\overline{\Omega}\times\mathcal{R}^{rec})$. (Let us note that without losing generality we can assume that $\beta>\alpha$.) Moreover,

13 Final remarks

13.1 Generalization to bigger dimensions

The obtained results can be extended to equations for the spatial sets $\Omega\subseteq\mathbb{R}^n$ with arbitrary $n<\infty$ and arbitrary number s of T variables, that is to say to equations of the form:

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \sum_{\kappa=1}^{s} \frac{\partial}{\partial T_\kappa} \left(\Gamma(T_\kappa)\, R\right) + f(t, x, T_1, \ldots, T_s). \tag{13.1}$$

Assumption 13.1. *Assume that* $\Gamma_1(T_1), \ldots, \Gamma_s(T_s)$ *are of* C^{k+1} *class,* $k \geq 2$*, and that for all* $(T_{10}, \ldots, T_{s0})$ *the system*

$$\frac{dT_1}{dt}(t) = \Gamma_1(T_1), \ldots, \frac{dT_s}{dt}(t) = \Gamma_s(T_s), \quad T_1(0) = T_{10}, \ldots, T_s(0) = T_{s0}. \tag{13.2}$$

has a unique C^{k+1} *solution* $(T_1(\cdot), T_8(\cdot))$ *satisfying the initial conditions* $T_1(0) = T_{10}$*, …,* $T_s(0) = T_{s0}$*, defined for all* $t \geq 0$*. Suppose that there exists a positive number* ρ_{1-s}*, such that*

$$\Gamma_1(T_1) \geq 0 \quad \textit{for } |T_1| \leq \rho_{1-s},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Gamma_s(T_s) \geq 0 \quad \textit{for } |T_s| \leq \rho_{1-s}.$$

Assumption 13.2. *Assume that for all* $x \in \overline{\Omega}$*,* $R_0(x, T_1, \ldots, T_s) \not\equiv 0$ *only for* $(T_1, \ldots, T_s)$ *from some open precompact set in* $\mathbb{R}^2_+$*.*

Let $\Omega = \mathbb{R}^n$ or let Ω be a bounded open subset of $\mathbb{R}^n$ with the boundary $\partial\Omega$ belonging to $C^{2+\upsilon}$ class, $\upsilon \in (0,1)$. Then the unique solution to the homogeneous version of Eq.(13.1) (with $f \equiv 0$), satisfying the homogeneous Neumann boundary conditions if Ω is bounded, can be expressed in the form

$$R(t, x, T_1, \ldots, T_s) = \int_\Omega G(t, x; 0, \xi) \mathcal{K}(T_1, \ldots, T_s; t) R_0(\xi, T_{10}(T_1, t), \ldots, T_{s0}(T_s, t)) d\xi\,.$$

Here G is either equal to G_0^n specified in Lemma 3.4 (if $\Omega = \mathbb{R}^n$), or G is equal to the Green's function for the homogeneous Neumann boundary value problem discussed in section 10, whereas

$$\mathcal{K}(T_1, \ldots, T_s; t) := \prod_{k=1}^{s} \frac{\Gamma_k(T_{k0}(T_k, t))}{\Gamma(T_k)}.$$

Next, the unique solution to the inhomogeneous problem with zero initial condition is equal to

$$u(t, x, T_1, \ldots, T_s) = \int_0^t \Big(\int_\Omega \mathcal{K}(T_1, \ldots, T_s; t-\tau) G(t, x; \tau, \xi) f(\tau, \xi, T_{10}(T_1, t-\tau), \ldots, T_{s0}(T_s, t-\tau)) d\xi \Big) d\tau.$$

If R_0 and f are of C^2 class and Γ_k, $k = 1, \ldots, s$ are of C^3 class of their arguments (in the corresponding domains), then the functions R and u are bounded in the norm of the space $C^{1+\upsilon/2, 2+\upsilon, 2}_{t, x, (T_1, \ldots, T_s)}([0,T] \times \overline{\Omega} \times \overline{\mathbb{R}^s_+})$. These solutions are unique in the space of functions $\mathcal{W}_{2B}$ which is defined similarly to the space $\mathcal{W}_2$ before Lemma 4.2 by replacing $\mathbb{R}^3$ with $\overline{\Omega}$.

13.2 Convolution notation

Let us note that the obtained expressions can be written in a bit more abstract form. Namely, if P denote the solution operator we have for the hyperbolic equation acting on the initial data function R_0, i.e. $P(R_0)(t, x, T_1, T_8)$ is a solution at time t. Then (3.52) can be written

$$R(t, x, T_1, T_8) = \int_\Omega G(t, x; 0, \xi) P(R_0)(t, \xi, T_1, T_8) d\xi =: G(t, x; 0, \xi) \circledast P(R_0)(t, \xi, T_1, T_8). \tag{13.3}$$

The last expression can be interpreted as a kind of convolution of the solution to the hyperbolic equation Eq.(3.3) with the Green's function G for the diffusion equation. Moreover, if G depends only on $x-\xi$, as it is for $\Omega = \mathbb{R}^n$ (see Lemma 3.4), then this expression is a usual convolution with respect to ξ, i.e.

$$R(t, x, T_1, T_8) = G(t, x; 0, \xi) \star_{\xi} P(R_0)(t, \xi, T_1, T_8). \tag{13.4}$$

Let us note that the above formula can be generalized to initial data at time $t = \tau > 0$. In this case, R_0 should be treated as a function of τ also, i.e.

$$R_0(\tau) = R_0(\tau, x, T_1, T_8).$$

Then the above equalities should be written as

$$R(t, x, T_1, T_8) = \int_{\Omega} G(t, x; \tau, \xi) P(R_0(\tau))(t, \xi, T_1, T_8) d\xi, \tag{13.5}$$

and

$$R(t, x, T_1, T_8) = G(t, x; \tau, \xi) \circledast P(R_0(\tau))(t, \xi, T_1, T_8), \tag{13.6}$$

where τ is fixed. In this notation

$$P(R_0(\tau))(t, \xi, T_1, T_8) = \mathcal{K}(T_1, T_8; t-\tau) \cdot R_0(\tau, \xi, T_{10}(T_1, t-\tau), T_{80}(T_8, t-\tau)).$$

Recall that $T_{10}(T_1, t-\tau)$ denotes the value of T_1 on the characteristic curve at time τ and $T_{80}(T_8, t-\tau)$ denotes the value of T_8 on the characteristic at time τ.

Let us note, that (13.3) and (13.4) are also valid, if $\Gamma = \Gamma(T_1, t)$ and $B = B(T_8, t)$, if P denotes the solution operator for Eq.(3.3). This follows from the first part of the proof of Lemma 3.52 and the fact that if $P(R_0)(t, \xi, T_1, T_8)$ satisfies Eq.(3.1), with $\Gamma = \Gamma(T_1, t)$ and $B = B(T_8, t)$, then

$$\Big(P(R_0)(t, \xi, T_1, T_8)\Big)_{,t} = -\frac{\partial}{\partial T_1}\Big(\Gamma(T_1, t) P(R_0)\Big) - \frac{\partial}{\partial T_8}\Big(B(T_8, t) P(R_0)\Big).$$

Now, as $G(t, x; 0, \xi)$ does not depend on T_1 and T_8, we conclude that

$$\begin{aligned} R(t, x, T_1, T_8) = &\int_{\Omega} \Big(G(t, x; 0, \xi) P(R_0)(t, \xi, T_1, T_8)\Big)_{,t} d\xi = \\ -&\frac{\partial}{\partial T_1}\Big(\Gamma(T_1, t) \int_{\Omega} G(t, x; 0, \xi) P(R_0)(t, \xi, T_1, T_8) d\xi\Big) \\ -&\frac{\partial}{\partial T_8}\Big(B(T_8, t) \int_{\Omega} G(t, x; 0, \xi) P(R_0)(t, \xi, T_1, T_8) d\xi\Big) = \\ -&\frac{\partial}{\partial T_1}\Big(\Gamma(T_1, t) R(t, x, T_1, T_8)\Big) - \frac{\partial}{\partial T_8}\Big(B(T_8, t) R(t, x, T_1, T_8)\Big). \end{aligned}$$

This fact is in agreement with the results of section 8.4. An example of a solution to Eq.(3.1) in the case of Γ and B depending also explicitly on t is given at the end of section 8.4.

Now, in the case of Γ and B not depending explicitly on t, as it follows from (3.56), the solution to the inhomogeneous equation (3.55) can be written as

$$R(t, x, T_1, T_8) = \int_{\Omega} \int_0^t G(t, x; \tau, \xi) P(f(\tau))(\tau, \xi, T_1, T_8) d\tau d\xi, \tag{13.7}$$

what can be displayed in the convolution form:

$$R(t, x, T_1, T_8) = G(t, x; \tau, \xi) \star P(f(\tau))(\tau, \xi, T_1, T_8).$$

It seems that in the case of Γ and B depending explicitly on t, equality corresponding to (13.7) do not hold. Similarly, in the case of Γ and B depending explicitly on x, we have not been able to derive the corresponding expressions for the solution even in the homogeneous case.

To obtain stronger results in the analysis of (1.1)-(1.3), in the remaining sections, we will propose a discrete time method, which can at least partially overcome these difficulties and study the existence of solutions to system (1.11)-(1.13).

Part III

Existence theorems via the Rothe method

14 Modified discrete Rothe method

The formulation of the discrete Rothe method for system (1.11)-(1.13) has been proposed in paper [19]. It was shown in [19] that, after some essential modifications taking into account the existence of characteristic curves for the hyperbolic counterpart in the first equation, the Rothe method can be used effectively to study the existence of solutions and well posedness of the initial boundary value problem.

As it was noticed in the previous section, system (1.11)-(1.13) can neither be studied by means of classical methods dedicated exclusively to systems of parabolic equations, nor by the methods dedicated exclusively to hyperbolic ones. The method proposed in [19] consists in a combined discretization of the variables t, T_1, T_8. In this setting, an implicit difference scheme exploits essentially in the form of the characteristic curves. The resulting sequence of elliptic PDE's is well posed and inherit, in a way, the basic properties of the original system (1.11)-(1.13).

Let $h = (\Delta t, \Delta T_1, \Delta T_8)$, $t^i = i\Delta t$, $T_1^j = j\Delta T_1$, $T_8^k = k\Delta T_8$ and

$$Z_h = \left\{ (t^i, T_1^j, T_8^k) : \ i, j, k = 0, 1, \dots \right\}, \quad Z_h' = \left\{ (T_1^j, T_8^k) : \ j, k = 0, 1, \dots \right\}.$$

Let $R^{i,j,k}(x) = R(t^i, x, T_1^j, T_8^k)$. Given any function $v : Z_h' \to R$, an interpolation operator I_h acting on v, can be defined informally by:

$$I_h v(T_1, T_8) = \text{ piecewise linear interpolation}$$

for $T_1 \in [T_1^{j-1}, T_1^j]$, $T_8 \in [T_8^{k-1}, T_8^k]$, $j, k = 1, 2, \dots$ In particular, we write below $I_h R^{i-1}(x; T_1, T_8)$ for the interpolation $I_h v$ where $v^{j,k} := R^{i-1,j,k}(x)$.

Thus the following numerical scheme for solving system (1.11)-(1.13) was proposed in [19]:

$$\begin{aligned} \frac{R^{i,j,k} - I_h R^{i-1}(x; \tau_1^{j,k}(x), \tau_8^{j,k}(x))}{\Delta t} &= d_R \nabla^2 R^{i,j,k} - \nabla \cdot (R^{i,j,k}\, \mathbf{K}^{i-1}(R^{i-1})) \\ -R^{i,j,k} &\left[\frac{\partial}{\partial T_1} \left(\widetilde{\gamma}(c_1^{u;i-1}, c_8^{u;i-1}, I_h T_1^j) \right) + \frac{\partial}{\partial T_8} \left(\widetilde{\delta}(c_8^{u;i-1}, I_h T_8^k) \right) \right] \end{aligned} \tag{14.1}$$

$$\frac{c_1^{u;i} - c_1^{u;i-1}}{\Delta t} = \nabla^2 c_1^{u;i} + \widetilde{\nu} \int_0^\infty \int_0^\infty I_h c_8^{8;i-1}\, I_h R^{i-1}\, dT_1\, dT_8 - c_1^{u;i} \tag{14.2}$$

$$\frac{c_8^{u;i} - c_8^{u;i-1}}{\Delta t} = \nabla^2 c_8^{u;i} + \widetilde{\mu} \int_0^\infty \int_0^\infty I_h c_1^{i-1,j}\, I_h R^{i-1}\, dT_1\, dT_8 - \widetilde{\pi}_8\, c_8^{u;i}. \tag{14.3}$$

where, for each $x \in \Omega$, $\tau_1^{j,k}, \tau_8^{j,k}$ are computed from the equations:

$$\frac{T_1^j - \tau_1^{j,k}(x)}{\Delta t} = \widetilde{\gamma}(c_1^{u;i-1}(x), c_8^{u;i-1}(x), T_1^j), \tag{14.4}$$

$$\frac{T_8^k - \tau_8^{j,k}(x)}{\Delta t} = \widetilde{\delta}(c_8^{u;i-1}(x), T_8^k). \tag{14.5}$$

Since $\dfrac{dT_1}{dt} \approx \dfrac{\Delta T_1}{\Delta t}$, $\dfrac{dT_8}{dt} \approx \dfrac{\Delta T_8}{\Delta t}$ the above equations are finite difference approximations of the equations determining the characteristics of the hyperbolic part of Eq.(1.11)

$$\frac{\partial R}{\partial t} = -\frac{\partial}{\partial T_1} \left(\tilde{\gamma}(c_1^u, c_8^u, T_1)\, R \right) - \frac{\partial}{\partial T_8} \left(\tilde{\delta}(c_8^u, T_8)\, R \right)$$

that is, by the equations:

$$\frac{dT_1}{dt} = \tilde{\gamma}(c_1^u, c_8^u, T_1) \tag{14.6}$$

$$\frac{dT_8}{dt} = \tilde{\delta}(c_8^u, T_8). \tag{14.7}$$

□

In section 16 we propose a different numerical scheme, where $x \in \overline{\Omega}$, T_1 and T_2 are treated as continuous variables. This change is dictated by the fact that we will concentrate mainly on the existence of solutions, putting aside the qualitative results of numerical simulations.

15 Preliminary lemmas and properties

In this section we establish some of the properties of the coefficient functions in system (1.11)-(1.13), and find some a priori estimates of its solutions.

15.1 Preliminary lemmas

Below, we will make use of the following formulation of the maximum principle for elliptic equations.

Lemma 15.1. *Let $\Omega \in \mathbb{R}^{m_\Omega}$, $m_\Omega \geq 1$, be a bounded domain with $\partial\Omega$ of $C^{2+\alpha}$ class, $\alpha \in (0,1)$. Let a_{ij}, $i,j \in \{1,\dots,m_\Omega\}$ and b_j, $j \in \{1,\dots,m_\Omega\}$ be of $C^{1+\alpha}(\overline{\Omega})$ class. Let*

$$Lw = \sum_{i,j=1}^{m_\Omega} a_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{j=1}^{m_\Omega} b_j(x)\frac{\partial w}{\partial x_j}$$

be a uniformly elliptic operator. Suppose that w satisfies the equation

$$Lw - c(x)w + f(x) = 0 \quad \text{in } \Omega, \tag{15.1}$$

$$\frac{\partial w}{\partial \mathbf{n}}(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{15.2}$$

where $c(x) > 0$ for $x \in \overline{\Omega}$ and f are non-negative in Ω. Then $w \geq 0$ in Ω and $w > 0$ in Ω unless $w \equiv 0$. In particular $w > 0$ in Ω, if $f \not\equiv 0$. Suppose that w attains a non-negative maximum at $x = x_0 \in \Omega$. Then

$$w(x_0) \leq \frac{f(x_0)}{c(x_0)}. \tag{15.3}$$

In general, for all $x \in \overline{\Omega}$,

$$0 \leq w(x) \leq \sup_{x\in\Omega}\Big[\frac{f(x)}{c(x)}\Big]. \tag{15.4}$$

□

Proof The proof follows from the fact that, thanks to boundary condition (15.2), the constant functions $\underline{w} \equiv 0$ and $\overline{w} \equiv \sup_{x\in\Omega}\Big[\frac{f(x)}{c(x)}\Big]$ are respectively sub and supersolution of Eq.(15.1) (see, e.g. [29, section 3.2]). The positivity of $w(\cdot)$ in $\overline{\Omega}$ follows from [29, Lemma 4.2, section 1.4]. Also, (15.3) follows from the proof of inequality (1.5) in [24, chapter III]. □

Below, we will also use the following generalization of Lemma 15.1 not assuming the non-negativity of the function f.

Lemma 15.2. *Let the assumptions of Lemma 15.1 be satisfied except for the assumption that $f(\cdot) \geq 0$. Then*

$$\inf_{y\in\Omega}\left[\frac{f(y)}{c(y)}\right] \le w(x) \le \sup_{y\in\Omega}\left[\frac{f(y)}{c(y)}\right]. \tag{15.5}$$

15.1.1 The uniqueness of solutions

Lemma 15.3. *Positive solutions to problem (15.1)-(15.2) are unique.*

Proof Let $r \in C^1(\Omega)$ and $r(x) > 0$ for $x \in \Omega$. Consider the eigenvalue problem:

$$(L - c(x) + \lambda r(x))\phi = 0 \tag{15.6}$$

for ϕ satisfying homogeneous boundary conditions. By means of Theorem 1.2 of chapter 3 in [29] and the assumption $c(x) > 0$ for $x \in \overline{\Omega}$, the principal eigenvalue λ^*, i.e. the eigenvalue $\lambda(r)$ satisfying Eq.(15.6) with the smallest real part, is real and positive $\lambda^* > 0$. Moreover, the corresponding eigenfunction ϕ^* is positive in Ω. Finally, using Theorem 3.2 of chapter 3 in [29], and noting that the upper solution and lower solutions can be taken in the form $K^* > 0$ and $(-K^*) < 0$ respectively satisfying

$$K^* \ge \sup_{x\in\Omega}\left[\frac{f(x)}{c(x)}\right].$$

we conclude that solutions to problem (15.1)-(15.2) are unique. □

Remark Lemma 15.3 can be also proved straightforwardly by means of the maximum principle applied to the homogeneous equation $Lw = 0$. (Any extremum cannot be attained at the boundary, unless w is constant, which for $c(x) \ge 0$, must be equal to zero. On the other hand and internal extremum must be also equal to zero.) □

Below, we will extensively use the following differentiability property for linear elliptic second order equations. This theorem is provided by general Schauder estimates in [1]. (See Theorem 7.3 for $\mathfrak{U} = \mathfrak{D}$.)

Lemma 15.4. *Suppose that Ω is bounded open subset of $\mathbb{R}^{m_\Omega}$, $m_\Omega \ge 1$, $l \ge 2$ and that $\partial\Omega \in C^{l+\beta}$ for some $\beta \in (0,1)$. Suppose that the coefficients of the elliptic operator*

$$\mathcal{L} = \sum_{i,j=1}^{m_\Omega} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{m_\Omega} b_j(x)\frac{\partial}{\partial x_j} - c(x)$$

have their $C^{(l-2)+\beta}$ norms bounded. Suppose that U satisfies the system

$$\mathcal{L}U = F(x) \quad \text{in } \Omega \tag{15.7}$$

$$n(x)\cdot\nabla U = \Phi(x) \quad \text{on } \partial\Omega. \tag{15.8}$$

Then U satisfies the estimate

$$\|U\|_{C^{l+\beta}(\Omega)} \le C_l\left(\|F\|_{C^{l-2+\beta}(\Omega)} + \|\Phi\|_{C^{l-1+\beta}(\partial\Omega)} + \|U\|_{C^0(\Omega)}\right).$$

The term $\|U\|_{C^0(\Omega)}$ can be omitted if the homogeneous problem has no nontrivial solutions.

Remark Similar estimate in the case of Dirichlet boundary conditions is given by inequality (1.11) of Section 1 of ch. 3 in [24]. □

An L_p version of the above property is given by the following lemma.

Lemma 15.5. *(Theorem 15.2 in [1]) Suppose that $l \ge 2$ and that $\partial\Omega \in C^{l+\beta}$ for some $\beta \in (0,1)$. Suppose that the coefficients of the elliptic operator $\mathcal{L}$ have their L^{l-2} norms bounded. Then, for any $p > 1$:*

$$\|U\|_{W_p^l} \le C_l\left(\|F\|_{W_p^{l-2}} + \|\Phi\|_{W^{l-1-(1/p)}(\partial\Omega)} + \|U\|_{C^0(\Omega)}\right),$$

where the constant C_l does not depend on the functions F, Φ and u_0. The term with $\|U\|_{C^0}$ can be omitted, if the corresponding homogeneous problem (with $F \equiv 0$) has no nontrivial solutions.

Similar property concerning linear parabolic equation is given by corresponding Schauder estimates.

Lemma 15.6. *(See, [23, Section IV, Theorem 5.3]) Let Ω is bounded open subset of $\mathbb{R}^{m_\Omega}$ and*

$$\mathcal{L} = \sum_{i,j=1}^{m_\Omega} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{m_\Omega} b_j(t,x)\frac{\partial}{\partial x_j} - c(t,x).$$

Let $T > 0$, $m > 0$ be a non-integer number, $\partial\Omega \in C^{m+2}$, and the coefficients of the operator $\mathcal{L}$ belong to the class $C^{m/2,m}(\Omega)$. Then, for any $f \in C^{m/2,m}(\Omega)$, and $\phi = U(o,x)$ and $\Phi \in C^{m+1}(\partial\Omega)$, which satisfy compatibility conditions of order $\left[\dfrac{m+1}{2}\right]$, the problem

$$\frac{\partial u}{\partial t} = \mathcal{L}u + f(t,x) \quad \text{in } (0,T)\times\Omega$$

$$u(0,x) = \phi(x) \quad \text{in } \Omega$$

$$n(x)\cdot\nabla u = \Phi(x) \quad \text{on } \partial\Omega$$

has a unique solution from $C^{1+m/2,2+m}(\Omega)$ satisfying the estimate

$$\|u\|_{C^{1+m/2,2+m}} \leq c\left(\|f\|_{C^{m/2,m}((0,T)\times\Omega)} + \|\phi\|_{C^{m+2}(\Omega)} + \|\Phi\|_{C^{m+1}(\partial\Omega)}\right),$$

where, for given $T > 0$, the constant c can be taken as independent of $t \in (0,T)$.

Remark In the case of homogeneous Neumann boundary conditions, that is to say, for $\Phi \equiv 0$, the compatibility conditions mentioned in Lemma 15.6, and defined explicitly before Theorem 5.1 in Section IV.5 of [23], are satisfied. □

16 Discrete-continuous numerical scheme for the simplified system

As we mentioned above, we propose a modified version of the numerical scheme (14.1)-(14.3). In this section, to get a preliminary insight into the properties of the considered system, we will put aside the proposed numerical method, discrete in the variables t, T_1 and T_8, and replace it by the iterative system (16.5)-(16.7). Instead of considering the complicated convective term, we will mimick it by adding appropriate terms proportional to R and the components of ∇R. We will assume that these terms are equal identically to zero close to the boundary of Ω (see Assumption 16.2). For simplicity, we will also change the notation and denote $\widetilde{\gamma}$ and $\widetilde{\delta}$ by γ and δ. Thus, below:

$$\begin{aligned}\gamma(c_1^u, c_8^u, T_1) &:= \left(\frac{2c_1^u}{\frac{c_1^u T_1}{c_1^u + f c_8^u + 1} + \widetilde{\widetilde{c}}_1} - \widetilde{\gamma}_2\right)\frac{T_1}{c_1^u + f c_8^u + 1} = \widetilde{\gamma}(c_1^u, c_8^u, T_1), \\ \delta(c_8^u, T_8) &:= 1 - \delta_2 \frac{T_8}{1 + c_8^u} = \widetilde{\delta}(c_8^u, T_8).\end{aligned} \tag{16.1}$$

For simplicity, we have also denoted $\delta_2 := \widetilde{\delta}_2$ in the definition of the function δ.

In this part, we will deal with the further simplified system, which differs from (1.11)-(1.13) only by replacement of the non-local (integral) term by a given function of R and the components of ∇R. We will thus consider the system:

$$\frac{\partial R}{\partial t} = d_R \nabla^2 R - \frac{\partial}{\partial T_1}\left(\gamma(c_1^u, c_8^u, T_1)\, R\right) - \frac{\partial}{\partial T_8}\left(\delta(c_8^u, T_8)\, R\right) - \Big(R \cdot F_1(t,x,T_1,T_8) - F_0(t,x)\cdot \nabla R\Big) \tag{16.2}$$

$$\frac{\partial c_1^u}{\partial t} = \nabla^2 c_1^u + \widetilde{\nu}\int_0^\infty \int_0^\infty c_8^8\, R\, dT_1\, dT_8 - c_1^u \tag{16.3}$$

$$\frac{\partial c_8^u}{\partial t} = \nabla^2 c_8^u + \widetilde{\mu}\int_0^\infty \int_0^\infty c_1\, R\, dT_1\, dT_8 - \widetilde{\pi}_8\, c_8^u. \tag{16.4}$$

To study the above system, we will use the following recurrence scheme:

$$d_R \nabla^2 R^i - \Big\{ R^i \left[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1((i-1)\Delta t, x, T_1, T_8) + \frac{1}{\Delta t}\right]$$
$$- \frac{R^{i-1}(x;\tau_1^{i-1}, \tau_8^{i-1})}{\Delta t} - F_0((i-1)\Delta t, x)\cdot \nabla R^i \Big\} \;=\; 0 \tag{16.5}$$

$$\frac{\partial c_1^{u;i}}{\partial t} = \nabla^2 c_1^{u;i} + \widetilde{\nu}\int_0^\infty \int_0^\infty c_{8*}^{8;i-1} T_8\, R^{i-1}\, dT_1\, dT_8 - c_1^{u;i} \quad \text{for } t \in ((i-1)\Delta t, i\Delta t] \tag{16.6}$$

$$\frac{\partial c_8^{u;i}}{\partial t} = \nabla^2 c_8^{u;i} + \widetilde{\mu}\int_0^\infty \int_0^\infty c_{1*}^{i-1} T_1\, R^{i-1}\, dT_1\, dT_8 - c_8^{u;i}\pi_8 \quad \text{for } t \in ((i-1)\Delta t, i\Delta t] \tag{16.7}$$

As in the previous scheme, for each $x \in \overline{\Omega}$, $\tau_1^{i-1}, \tau_8^{i-1}$ are computed from the equations:

$$\frac{T_1 - \tau_1^{i-1}}{\Delta t} = \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1), \tag{16.8}$$

$$\frac{T_8 - \tau_8^{i-1}}{\Delta t} = \delta(c_{8*}^{u;i-1}(x), T_8). \tag{16.9}$$

For all the considered $i \geq 0$, we assume that the functions R^i, $c_1^{u;i}$ and $c_8^{u;i}$ satisfy, according to a (1.5) and (1.6), the homogeneous Neumann boundary conditions.

In Eq.(16.5), $c_{1*}^{u;0}(x) = c_{10}^u(x)$, $c_{8*}^u(x) = c_{80}^u(x)$, whereas for $i \in \{1, \ldots, n+1\}$ we denoted:

$$c_{1*}^{u;i-1}(x) := c_1^u((i-1)\Delta t, x), \quad c_{8*}^{u;i-1}(x) := c_8^u((i-1)\Delta t, x). \tag{16.10}$$

In Eqs (16.6)-(16.9), for $i \in \{1, 2, \ldots, n+1\}$ we denoted:

$$c_{1*}^{i-1}(x) := \frac{c_{1*}^{u;i-1}(x)}{1 + f c_{8*}^{u;i-1}(x) + c_{1*}^{u;i-1}(x)}, \quad c_{8*}^{8;i-1}(x) := \frac{c_{8*}^{u;i-1}(x)}{1 + c_{8*}^{u;i-1}(x)}. \tag{16.11}$$

Eqs.(16.6),(16.7) are solved sequentially on each of the interval $[(i-1)\Delta t, i\Delta t)$ by assuming the initial conditions at $t = (i-1)\Delta t$:

$$c_1^{u;i}((i-1)\Delta t, x) = c_1^{u;i-1}((i-1)\Delta t, x) = c_{1,*}^{u;i-1}(x), \quad c_8^{u;i}((i-1)\Delta t, x) = c_8^{u;i-1}((i-1)\Delta t, x) = c_{8,*}^{u;i-1}(x). \tag{16.12}$$

16.1 Main assumptions

Below, we will suppose that the following conditions are satisfied.

Assumption 16.1. *Ω is a bounded domain (open and connected) in $\mathbb{R}^3$, with the boundary $\partial\Omega$ of $C^{3+\beta}$ class with $\beta \in (0,1)$.*

Remark In general, the analysis which is carried out below hold also for $\Omega \subset \mathbb{R}^{m_\Omega}$, with $m_\Omega \geq 1$. $\square$

Assumption 16.2. *The function $F_1 : [0,T] \times \overline{\Omega} \times \mathbb{R}^2 \mapsto \mathbb{R}$ is of C^1 class with respect to t and all of its derivatives up to the order of 4 with respect to the components of x and (T_1, T_8) are continuous and bounded. Let $\|F_1\|$ denote the sum of the suprema of $|F_1|$, $|F_{1,t}|$ and all the derivatives with respect to the components of x and (T_1, T_8) up to order 4. Let*

$$f_1 := \|F_1\|.$$

The function $F_0 = (F_{01}, F_{02}, F_{03}) : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^3$ is of $C^{t,x}_{1,4}([0,T])$ class. Let

$$f_0 := \|F_0\|_{C^{t,x}_{1,4}([0,T]\times\overline{\Omega}}.$$

There exists a number $\delta > 0$ such that $F_1(t, x, T_1, T_8) \equiv 0$, $F_0(t,x) \equiv 0$ for all $(t, T_1, T_8) \in [0,T] \times \mathbb{R}^2$, if only $dist(x, \partial\Omega) < \delta$.

Note that the last assumption is in accordance with the cutting off properties of the function Ψ in definition (1.7). Additionally, we assumed that the function F_0 does not depend on T_1, T_8. This assumption significantly simplifies the problem of obtaining 'a priori' estimates.

In our analysis, we will fix finite $T > 0$ and consider the above scheme for $t \in [0,T]$, and $i \in \{1, n\}$, with n sufficiently large, and Δt satisfying the condition

$$T = n\Delta t. \tag{16.13}$$

It means that n depends on Δt, $n = n(\Delta t) = \dfrac{T}{\Delta t}$.

Remark The right hand side of Eqs (16.6)-(16.7) depend on the function R^{i-1} which, in general, is discontinuous as a function of the index $i-1$. However, on each of the open set $((i-1)\,\Delta t, i\Delta t)$ one can treat these equations as a system of two parabolic equations depending in a non-local way on the function $R^{i-1}(x)$, which is smooth with respect to $x \in \Omega$. $\square$

Remark Due to the uniqueness of solutions to the considered parabolic initial boundary value problems, we can treat Eqs (16.6)-(16.7) as defined on the whole of the time interval $[0,T]$. In this approach, let us define:

$$c_1^u(t,x) := \sum_{i=1}^{n} \chi_i c_1^{u;i}(t,x), \quad c_8^u(t,x) := \sum_{i=1}^{n} \chi_i c_8^{u;i}(t,x) \tag{16.14}$$

where χ_i denotes the characteristic function of the interval $[(i-1)\Delta t, i\Delta t)$. $\square$

From the definition (16.8)-(16.9) we can extract a simple fact, which will be the basis of our estimates below.

Lemma 16.3. *Suppose that the functions $c_{1*}^{u;i-1}(\cdot)$ and $c_{8*}^{u;i-1}(\cdot)$ are of $C^1(\overline{\Omega})$ class. Then, for $k \in \{1,2,3\}$, $i = \{2, \ldots, n+1\}$ and all $\Delta t > 0$, we have for each fixed T_1 and T_8:*

$$\frac{\partial \tau_1^{i-1}}{\partial x_k} = -\frac{\partial (T_1 - \tau_1^{i-1})}{\partial x_k} = -\frac{\partial \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial x_k} \cdot \Delta t,$$

$$\frac{\partial \tau_8^{i-1}}{\partial x_k} = -\frac{\partial (T_8 - \tau_8^{i-1})}{\partial x_k} = -\frac{\partial \delta(c_{8*}^{u;i-1}(x), T_8)}{\partial x_k} \cdot \Delta t.$$

These derivatives are thus of the order $O(\Delta t)$ as $\Delta t \to 0$ and of class $C^0(\overline{\Omega})$.

Likewise, if the functions $c_{1*}^{u;i-1}(\cdot)$ and $c_{8*}^{u;i-1}(\cdot)$ are of $C^p(\overline{\Omega})$ class, then the p-th order derivatives of τ_m^{i-1} with respect to the components of x are of class $C^0(\overline{\Omega})$ and are products of Δt and functions independent of Δt.

Our analyses are based on the assumption of smoothness of the initial data as well as of the compactness of the initial data with respect to the variables T_1 and T_8. In reference to system (16.5)-(16.7), this assumption can be expressed in the following form.

As in section 3 (see (3.2)), let us define

$$\overline{\mathbb{R}^2_+} := \{(r_1, r_8) \in \mathbb{R}^2 : r_1 \geq 0,\, r_8 \geq 0\}.$$

Assumption 16.4. *Suppose that:*

1. $R^0 \in C^4(\overline{\Omega} \times \overline{\mathbb{R}^2_+})$ is compactly supported, with the support contained in $\overline{\Omega} \times [0, T^0_{1*}] \times [0, T^0_{8*}]$ and that R_0 satisfies conditions (1.18)

2. $c_1^{u;0}, c_8^{u;0} \in C^4(\overline{\Omega})$

3. $R^0(x, T_1, T_8) \geq 0$, $c_1^{u;0}(x), c_8^{u;0}(x) \geq 0$, for all $(x, T_1, T_8) \in \partial\Omega \times \overline{\mathbb{R}^2_+}$.

Remark The smoothness demands of the initial data R^0, $c_1^{u;0}$, $c_8^{u;0}$ are implied by the method of obtaining a priori estimates used below, which are established by consecutive differentiation. □

16.2 Description of the method of proving the existence of solutions to a variation of system (1.11)-(1.13)

The solutions to system (16.5)-(16.7), (16.8)-(16.9) will be used to obtain solutions to system (1.11)-(1.13) with the term $\nabla \cdot (R\,\mathbf{K}(R))$ replaced by the term $(R \cdot F_1(t, x, T_1, T_8) - F_0(t, x) \cdot \nabla R)$. By this replacement we fix our attention on the problems connected with the lack of diffusion terms of the variables T_1 and T_8, and put aside the difficulties connected with the term $\nabla \cdot (R\,\mathbf{K}(R))$. These issues have been undertaken and, at least partially solved, for a scalar equation corresponding to Eq.(1.11) in the papers [9], [10], where *in contrast* the hyperbolic like terms $\dfrac{\partial}{\partial T_1}(\gamma\, R)$ and $\dfrac{\partial}{\partial T_8}(\delta\, R)$ are not present. Below, by studying the properties of the numerical scheme (16.5)-(16.7), (16.8)-(16.9), we will be interested in establishing the existence classical solutions to system (16.2)-(16.4).

The method of proving the existence of solutions to system (16.2)-(16.4) is based on deriving a series of a priori estimates for solutions to system (16.5)-(16.7), (16.8)-(16.9), i.e. the functions R^i, $c_1^{u;i}$, $c_8^{u;i}$ in the spaces of differentiable functions. According to this, we estimate the derivatives of the functions R^i both with respect to the components of the space variable x as well as with respect to T_1 and T_8. These estimates stay bounded for all i and keep their validity for $\Delta t \to 0$.

In the preliminary step we modify the function γ in the region $T < 0$. The objective of such a modification is to guarantee that the support of the functions R^i does not contain points (x, T_1, T_8) with negative values of T_1.

Thus in section 16.5 we establish a priori bounds of the absolute values of the functions R^i. These bounds can be found due the appropriate structure of the function δ and the modified function γ, implying agreeable properties of their derivatives (examined and listed in Lemma 16.7). An additional assumption necessary to establish the bounds for R^i is the non-negativity of the functions $c_1^{u;i}$, $c_8^{u;i}$. However, this feature is inherited at every step of the iterative sequence, so is implied by the initial data. In the same section, using the properties of the functions γ and δ, we find the bounds for the increase of the support of the functions R^i with respect to (T_1, T_8) (see Lemma 16.9). In the next step, we examine differential properties of the functions c_1^u and c_8^u defined in (16.45) as functions of $t \in [0, T]$ and $x \in \overline{\Omega}$. Interestingly enough, these functions are of $C^{(1+\beta)/2, 1+\beta}_{t,x}$ class, i.e. they are Hölder continuous in t with exponent $(1+\beta)/2$, $\beta \in (0, 1)$, and have continuous in t first derivatives with respect to x (see Lemma 16.11). Having the uniform (with respect to i) boundedness of the functions $c_1^{u;i}$, $c_8^{u;i}$ in $C^1(\Omega)$ norm, which can be obtained only on the condition that $\|R^i\|_{C^0}$ is uniformly bounded, we can establish the uniform boundedness of the derivatives of the functions R^i. In section 16.8 we find the estimates for the first derivatives of R^i with respect to T_l, $l = 1, 8$, in section 16.11 the second order derivatives $R^i_{T_l T_m}$, whereas in section 16.12 for the third order derivatives $R^i_{T_l T_m T_p}$. In sections 16.9 and 16.10, we obtain the estimate for the first order derivatives of R^i with respect to the components of x. In section 16.14 we find a priori estimates for the mixed second order derivatives of the form $R^i_{x_k T_m}$. The bounds of the first derivatives $R^i_{x_k}$ and $R^i_{T_l}$ allow us to prove Lemma 16.12. By means of these estimates, in section 16.15, we are able to analyse the difference between the functions corresponding to subsequent values of i. To be more precise, we analyse the functions $Z^i = R^i - R^{i-1}$ and the functions $H^i_j = R^i_{T_j} - R^{i-1}_{T_j}$. This result empowers us to demonstrate, in section 16.16, the uniform with respect to i boundedness of $C^{1+\beta}_x$ norms of the functions R^i. Using the last conclusion, we show in section 16.17 the higher order differentiability of the functions $c_1^{u;i}$ and $c_8^{u;i}$, in particular the fact that the differences between the corresponding derivatives of these functions with respect to the components of x on adjacent intervals, i.e. $[(i-1)\Delta t, i\Delta t]$ and $[i\Delta t, (i+1)\Delta t]$ are of the order of $O(\Delta t)$. This finding is crucial to obtaining in section 16.18, estimates of first order derivatives of Z^i with respect to x_k, together with the differences of the mixed second derivatives $R^i_{x_k T_j} - R^{i-1}_{x_k T_j}$, and bounds for the $C^{2+\beta}_x$ norm of R^i in section 16.19. In the same section we use the refined version of the Gagliardo-Nirenberg inequality from [4] and obtain additionally some Hölder estimates for the derivatives of the functions Z^i. Finally in section 17, using the functions R^i, c_1^u and c_8^u, we construct an approximate solution to system (16.2)-(16.4) and consider its convergence to a classical solution as $\Delta t \to 0$.

The method of the existence proof can thus be displayed schematically in a graphical form as below.

1 Subsection 16.3. Modification of the function γ aimed to guarantee that $R(t, x, T_1, T_8) \equiv 0$ in the region $\{(T_1, T_8) : T_1 < 0 \vee T_8 < 0\}$ (proved in lemma 16.5).

2 Subsection 16.4. A priori bounds of the functions γ and δ and their derivatives established in Lemma 16.6 and Lemma 16.7.

3 Subsection 16.5. Estimates for the upper bound of the function R^i for sufficiently small $\Delta t > 0$.
(Based on **2**.)

4 Subsection 16.5. Estimates of the support of R^i with respect to (T_1, T_8).
(Based on **2** and **3**.)

5 Subsection 16.7. Estimates for the upper bounds of the $C^{(1+\beta)/2, 1+\beta}_{t,x}$ norms of the functions

$$c_1^u(t,x) := \sum_{i=1}^{n} \chi_i c_1^{u;i}(t,x), \text{ and } c_8^u(t,x) := \sum_{i=1}^{n} \chi_i c_8^{u;i}(t,x)$$

(Lemma 16.11). (Based on **3** and **4**.)

6 Subsection 16.8. A priori estimates of the first derivatives of the function $R^i(x, T_1, T_8)$ with respect to T_m, $m = 1, 8$.

7 Subsection 16.9. Interior estimates of the first derivatives of the function $R^i(x, T_1, T_8)$ with respect to x_r, $r = 1, 2, 3$.

8 Subsection 16.10. Estimates of the first derivatives of the function $R^i(x, T_1, T_8)$ with respect to x_r, $r = 1, 2, 3$, at the boundary of Ω. (Based on **6**.)

9 Subsections 16.11 and 16.12. Estimates of the second and third order derivatives of functions $R^i(x, T_1, T_8)$ with respect to T_1 and T_8. (Based on **6**.)

10 Subsection 16.13. Estimates of the mixed second order derivatives of function $R^i(x, T_1, T_8)$ with respect to x_r and T_m for $r = 1, 2, 3$ and $m \in \{1, 8\}$. (Based on **6**, **7** and **8**.)

11 Subsection 16.14. Estimates for the mixed third order derivatives of $R^i_{x_r T_l T_m}(x, T_1, T_8)$.

12 Subsection 16.15. Estimates of the differences Z_i between the functions R^i corresponding to subsequent values of i.

13 Subsection 16.16. Estimates of $C^{1+\beta}_x$ norms of the functions R^i:

$$\|R^i\|_{C^{1+\beta}(\Omega)} \le C_{1\beta}.$$

(Based on **12**.)

14 Subsection 16.17. Estimates of higher order derivatives of the functions $c_1^{u;i}$ and $c_8^{u;i}$ on the subintervals $[(i-1)\Delta t, i\Delta t]$. (Based on **13**.)

15 Subsection 16.18. Estimates of the first derivatives of the functions Z^i with respect to x_k.

16 Subsection 16.19. Estimates of $C^{2+\beta}_x$ norms of the functions R^i:

$$\|R^i\|_{C^{2+\beta}(\Omega)} \le C_{2\beta}.$$

These estimates enable us to use the refined version of Gagliardo-Nirenberg inequality to obtain Hölder estimates for the derivatives of the functions Z^i. (Based on **15**.)

17 Subsection 16.20. Estimates of the differences $Z^i - Z^{i-1}$ via a version of the Gagliardo–Nirenberg interpolation inequality. (Based on **12** and **15**.)

18 Section 17. Convergence of the approximate solutions to solutions to system (16.2)-(16.4).

16.3 Modification of the function γ

To begin with, let us note that, from the biological point of view, the probability of finding cells characterized by negative values of T_1 and T_8 should be identically equal to zero, i.e. $R(t, x, T_1, T_8) = 0$ for $(t, x) \in [0, T] \times \overline{\Omega}$, if only $T_1 < 0$ or $T_8 < 0$. In general, the support of R with respect to T_1 and T_8 can change during the evolution, and after some time comprise points with negative values of T_1 or T_8, even if such points are outside the support of R for $t = 0$.

According to the form of the function δ, for $\Delta t \ge 0$ and all $c_{8*}^{u;i-1}$ such that $\overline{c}^u_{8*} \le c_{8*}^{u;i-1}(x) \ge 0$ for $x \in \Omega$, we have, for each $i \in \{2, \dots, n\}$,

$$\tau_8^{i-1}(c_{8*}^{u;i-1}(x), T_8) < T_8 - \Delta t + \Delta t \frac{T_8\ \delta_2}{1 + \overline{c}^u_{8*}} = T_8\Big(1 + \Delta t \frac{\delta_2}{1 + \overline{c}^u_{8*}}\Big) - \Delta t$$

hence

$$\tau_8^{i-1}(x, T_8) < 0$$

if $T_8 \le 0$ independently of $\Delta t \ge 0$. As it will be shown in the proof of Lemma 16.5, this inequality implies that $R^{i-1}(x, T_1, T_8) \equiv 0$ for all $T_8 < 0$, if only $R^0(x, T_1, T_8)$ satisfies the same condition.

However, to guarantee that our numerical scheme implies the similar property with respect to the variable T_1, we will consider system (16.5)-(16.7) with appropriately modified function γ.

Let

$$\gamma_* = \gamma \cdot \Psi_\gamma(T_1),$$

where

$$\Psi_\gamma(T_1) := \begin{cases} 0 & T_1 \in (-\infty, -1/2\tilde{\tilde{c}}_1] \\ \dfrac{\Psi_*(T_1 + 1/2\tilde{\tilde{c}}_1)}{\Psi_*(T_1 + 1/2\tilde{\tilde{c}}_1) + \Psi_*(-1/4\tilde{\tilde{c}}_1 - T_1)} & T_1 \in (-1/2\tilde{\tilde{c}}_1, -1/4\tilde{\tilde{c}}_1) \\ 1 & T_1 \geq -1/4\tilde{\tilde{c}}_1 \, , \end{cases} \tag{16.15}$$

and $\Psi_*(s)$ is given after (1.8). As a result, the function γ_* is smooth everywhere in the region $\{(c_1^u, c_8^u, T_1) \,:\, c_1^u \geq 0, c_8^u \geq 0, T_1 \in \mathbb{R}\}$. Next, we will show that for such a modified function $\gamma = \gamma_*$, we can find a global estimate of the maximal values of the functions $c_1^u(t,x)$ and $c_8^u(t,x)$ for $(t,x) \in [0,T] \times \overline{\Omega}$. This will imply that the function γ_* is bounded for all $(t,x,T_1) \in [0,T] \times \overline{\Omega} \times \mathbb{R}$. Consequently, in this set

$$\tau_1^{i-1}(x, T_1) = T_1 - \gamma_*(x, T_1)\Delta t, \tag{16.16}$$

where by $\gamma_*(x, T_1)$ we denoted $\gamma_*(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)$. Due to the form of γ_*, $\gamma_*(x, T_1 = 0) = 0$

$$\tau_1^{i-1}(x, T_1) - T_1 = -\gamma_{*,T_{1*}} T_1 \, \Delta t,$$

where $T_{1*} \in (0, T_1)$ for $T_1 > 0$. Now, due to the estimates provided by Lemma 16.7 (see (16.22)) below the derivative of γ_* with respect to T_1 is uniformly bounded, i.e. $|-\gamma_{*,T_1}| < C_{1\gamma}$, thus for $\Delta t < \dfrac{1}{2C_{1\gamma}}$ we have for $T_1 \geq 0$:

$$\tau_1^{i-1}(x, T_1) > T_1 - \frac{1}{2}T_1 = \frac{1}{2}T_1 \geq 0. \tag{16.17}$$

Likewise, for $T_1 < 0$, we have for $\Delta t < \dfrac{1}{2C_{1\gamma}}$ by means of (16.16):

$$\tau_1^{i-1}(x, T_1) < T_1 + \frac{1}{2}|T_1| = \frac{1}{2}T_1 < 0. \tag{16.18}$$

It follows that for $\Delta t > 0$ sufficiently small the regions $T_1 > 0$ and $T_1 < 0$ do not mix under the action of system (16.8)-(16.9). In view of the above, the following lemma holds.

Lemma 16.5. *Suppose that for all $i \in \{0, 1, \ldots, n\}$ and all $(t,x) \in [0,T] \times \overline{\Omega}$ the functions $c_1^{u;i}(t,x)$ and $c_8^{u;i}(t,x)$ are non-negative and uniformly bounded in their absolute value by a (finite) constant. Suppose that $R^0(x, T_1, T_8) = 0$ in the region $\{T_1 < 0\} \cup \{T_8 < 0\}$. Then, for $\Delta t \geq 0$ sufficiently small,*

$$R^i(x, T_1, T_8) = 0 \quad \textit{for all } i \in \{1, \ldots, n\} \textit{ in the region } \{T_1 < 0\} \cup \{T_8 < 0\}. \tag{16.19}$$

Proof The proof follows by induction. Thus, suppose that, for $i \in \{1, \ldots, n-1\}$, $R^{i-1}(x, T_1, T_8) \equiv 0$ in the set $\{T_1 < 0\} \cup \{T_8 < 0\}$, hence by what was noted above, in particular, by (16.18), $R^{i-1}(x, \tau_1^{i-1}(x, T_1), \tau_1^{i-1}(x, T_8)) = 0$. Then, by means of estimate (15.4) in Lemma 15.1 and the subsection 15.1.1, $R^i(x, T_1, T_8) \equiv 0$ in the region $\{T_1 < 0\} \cup \{T_8 < 0\}$. □

Lemma 16.5 implies a strategy to guarantee that $R(t, x, T_1, T_8) \equiv 0$ in the region $\{(T_1, T_8) : T_1 < 0 \vee T_8 < 0\}$. Then, due to the fact that the modification of γ takes place in the region $\{T_1 < 0\}$, we will be able to conclude that we have obtained a solution for the system with non-modified function γ.

16.4 Properties of the function δ and the modified function γ_*

As it is seen from Eq. (1.11), crucial for the analysis of the considered system are the properties of the functions γ_* and δ.

Lemma 16.6. *The values of the functions γ and δ are bounded from above and below for any compact subset of the set $\{(T_1, T_8, c_1^u, c_8^u) : (T_1, T_8) \geq 0,\ (c_1^u, c_8^u) \geq 0\}$. Given the values of c_1^u and c_8^u*

$$\delta(c_8^u, T_8) < 0 \quad \text{for} \quad \delta_2 T_8 > (1 + c_8^u).$$

Similarly,

$$\gamma_*(c_1^u, c_8^u, T_1) < 0 \quad \text{for } T_1 > \max\left\{0, \frac{(2c_1^u - \widetilde{\widetilde{c}}_1\gamma_2)(c_1^u + fc_8^u + 1)}{c_1^u\gamma_2}\right\}.$$

Finally, for all $(T_1, T_8, c_1^u, c_8^u) \in \{(T_1, T_8, c_1^u, c_8^u) : (T_1, T_8) \geq 0,\ (c_1^u, c_8^u) \geq 0\}$, we have:

$$\gamma_*(c_1^u, c_8^u, T_1) \leq 2 \quad \text{and} \quad \delta(c_8^u, T_8) \leq 1. \tag{16.20}$$

Next, the following lemma holds.

Lemma 16.7. *For $c_8^u \geq 0$*

$$\frac{\partial}{\partial T_8}\delta(c_8^u, T_8) = -\delta_2 \frac{1}{1 + c_8^u} < 0. \tag{16.21}$$

The partial derivative

$$\frac{\partial}{\partial T_1}\gamma_*(c_1^u, c_8^u, T_1)$$

is bounded from above and below uniformly with respect to $(T_1, T_8) \in \mathbb{R}^2$ and $(c_1^u, c_8^u) \geq (0, 0)$, i.e. there exist finite positive constants A_- and A_+ such that

$$-A_- \leq \frac{\partial}{\partial T_1}\gamma_*(c_1^u, c_8^u, T_1) \leq A_+ \tag{16.22}$$

and

$$-A_- \leq \frac{\partial^2}{\partial T_1^2}\gamma_*(c_1^u, c_8^u, T_1) \leq A_+. \tag{16.23}$$

Also,

$$\begin{aligned} -\delta_2\, T_8 &\leq \frac{\partial^2}{\partial c_8^u}\delta(c_8^u, T_8) < 0, \\ -\delta_2 &\leq \frac{\partial^2}{\partial c_8^u \partial T_8}\delta(c_8^u, T_8) < 0, \end{aligned} \tag{16.24}$$

and there exists a non-negative constant $\mathcal{M}$, depending on $\widetilde{\widetilde{c}}_1$, f, and γ_2, such that

$$-\mathcal{M}T_1 \leq \frac{\partial}{\partial c_1^u}\gamma_*(c_1^u, c_8^u, T_1),\ \frac{\partial}{\partial c_8^u}\gamma_*(c_1^u, c_8^u, T_1) \leq \mathcal{M}T_1 \tag{16.25}$$

$$-\mathcal{M} \leq \frac{\partial^2}{\partial c_1^u \partial T_1}\gamma_*(c_1^u, c_8^u, T_1),\ \frac{\partial^2}{\partial c_8^u \partial T_1}\gamma_*(c_1^u, c_8^u, T_1) \leq \mathcal{M} \tag{16.26}$$

together with

$$\begin{aligned} -\mathcal{M} &\leq \frac{\partial^3}{\partial c_k^u \partial c_m^u \partial T_1}\gamma_*(c_1^u, c_8^u, T_1) \leq \mathcal{M}, \quad k, m = 1, 8, \\ -\mathcal{M}T_1 &\leq \frac{\partial^2}{\partial c_k^u \partial c_m^u}\gamma_*(c_1^u, c_8^u, T_1) \leq \mathcal{M}T_1, \quad k, m = 1, 8, \end{aligned} \tag{16.27}$$

and

$$-2 \leq \frac{\partial^3}{\partial c_8^u \partial c_8^u \partial T_8} \delta(c_8^u, T_8) < 0. \tag{16.28}$$

for all non-negative T_1, T_8, c_1^u and c_8^u.

Proof The proof follows from straightforward differentiation. For example, we have:

$$\frac{\partial}{\partial T_1} \gamma_*(c_1^u, u_3, T_1) = \frac{2c_1^u \widetilde{\widetilde{c}}_1 (c_1^u + f c_8^u + 1)}{(\widetilde{\widetilde{c}}_1 (c_1^u + f c_8^u + 1) + c_1^u T_1)^2} - \gamma_2 \frac{1}{c_1^u + f c_8^u + 1}$$

from where follow the first two claims of the lemma. The remaining statements are proven in the similar way. □

16.5 Estimate of the upper bound of the functions R^i

Let us start from deriving an estimate of the norm $\|R^i\|_{L^\infty}$, $i = 1, \ldots, n$, where

$$\|R^i\|_{L^\infty} = \sup_{x \in \overline{\Omega}, T_1 \in \mathbb{R}, T_8 \in \mathbb{R}} |R^i(x, T_1, T_8)|.$$

These estimates will be obtained by means of Lemma 15.1, hence the *sine qua non* property of system (16.5)-(16.7) allowing for their establishing is the boundedness of the expression

$$\left[\frac{\partial}{\partial T_1} \left(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) \right) + \frac{\partial}{\partial T_8} \left(\delta(c_{8*}^{u;i-1}, T_8) \right) + F_1((i-1)\Delta t, x, T_1, T_8) \right]$$

for all $(t, x) \in [0, T] \times \overline{\Omega}$ and all non-negative values of $c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1, T_8$.

Remark Below, for simplicity, we will denote the function γ_* by γ. □

Lemma 16.8. *For all sufficiently small $\Delta t > 0$ and all $i \in \{1, \ldots, n\}$*

$$\|R^i\|_{L^\infty} \leq \|R^0\|_{L^\infty} \frac{1}{\left((1 - A\frac{T}{n})^n\right)^{\frac{i}{n}}} < \frac{3}{2} \|R^0\|_{L^\infty} \exp(Ai\Delta t), \tag{16.29}$$

where

$$A := -A_- - \delta_2 - f_{1-} \tag{16.30}$$

with A_- and δ_2 defined in Lemma 16.7 and

$$f_{1-} := \inf_{t \in [0,T], x \in \overline{\Omega}, T_1 \in \mathbb{R}, T_8 \in \mathbb{R}} F_1(t, x, T_1, T_8).$$

Proof By means of Lemma 16.7 and the fact that the parameter Δt can be taken sufficiently small, we can derive, using Lemma 15.1, the following recurrence inequalities:

$$\|R^i\|_{L^\infty} \leq \frac{\|R^{i-1}\|_{L^\infty}/\Delta t}{1/\Delta t - A}$$

hence

$$\|R^i\|_{L^\infty} \leq \frac{\|R^{i-1}\|_{L^\infty}}{1 - A\Delta t}.$$

By composing the above inequalities for $i \leq n$, we obtain

$$\|R^i\|_{L^\infty} \leq \frac{\|R^0\|_{L^\infty}}{(1 - A\Delta t)^i}.$$

Thus for $i = n$ and with n sufficiently large

$$\|R^n\|_{L^\infty} \leq \frac{\|R^0\|_{L^\infty}}{(1-A\Delta t)^n} = \frac{\|R^0\|_{L^\infty}}{(1-A\dfrac{T}{n})^n} < \frac{3}{2}\,\|R^0\|_{L^\infty}\exp(An\Delta t). \tag{16.31}$$

It follows that for $\Delta t > 0$ sufficiently small:

$$\|R^i\|_{L^\infty} \leq \|R^0\|_{L^\infty}\frac{1}{\left((1-A\frac{T}{n})^n\right)^{\frac{i}{n}}} < \frac{3}{2}\,\|R^0\|_{L^\infty}\exp(Ai\Delta t). \tag{16.32}$$

□

Remark Let $\mathcal{N}_A := AT$ and let, for $\mathbb{N} \ni n > \lceil \mathcal{N}_a \rceil$, $\mathbb{R} \ni \kappa_n := n/\mathcal{N}_A$, where $\lceil \mathcal{N}_A \rceil$ denotes the least integer that is greater than or equal to $\mathcal{N}_A$. Then

$$\frac{1}{(1-\frac{AT}{n})^n} = \left[\frac{1}{\left(1-\frac{1}{\kappa_n}\right)^{\kappa_n}}\right]^{\mathcal{N}_A}.$$

It follows that to analyse the left hand side of this relation it suffices to consider the sequence inside the square bracket at the right hand side. We have

$$\log\left(\frac{1}{(1-\frac{1}{y})^y}\right) = \log\left(\frac{y^y}{(y-1)^y}\right) = y\log\left(1+\frac{1}{y-1}\right).$$

As the Taylor expansion of $\log(1+z)$ around $z=0$ is an alternating series with the first element equal to z, then, according to the Leibniz theorem for alternating series, its sum is smaller than z. It follows that the last expression is smaller than $\dfrac{y}{y-1}$. It follows that for every $\mathcal{W} > 1$ there exists y so large that

$$\frac{y}{y-1} \leq \log(\mathcal{W}+\exp(1)) = \log(\mathcal{W}) + 1.$$

This holds for $y \geq 1 + 1/log(\mathcal{W})$. Consequently, for $\kappa_n \geq 1 + 1/log(\mathcal{W})$

$$\frac{1}{\left(1-\dfrac{1}{\kappa_n}\right)^{\kappa_n}} \leq \mathcal{W}\,\exp(1),$$

and

$$\left[\frac{1}{\left(1-\dfrac{1}{\kappa_n}\right)^{\kappa_n}}\right]^{\mathcal{N}_A} \leq \mathcal{W}^{\mathcal{N}_A}\,\exp(\mathcal{N}_A).$$

Suppose that

$$\mathcal{W}^{\mathcal{N}_A} \leq m_A. \tag{16.33}$$

(In our choice $m_A = 3/2$.) Then $\mathcal{W} \leq (m_A)^{\frac{1}{\mathcal{N}_A}}$ and $log(\mathcal{W}) \leq log(m_A)/\mathcal{N}_A$, hence

$$\kappa_n \geq 1 + \frac{\mathcal{N}_A}{log(m_A)},$$

so

$$n = \left\lceil \mathcal{N}_A \cdot \kappa_n \right\rceil \geq \mathcal{N}_A \cdot \left(1+\frac{\mathcal{N}_A}{log(m_A)}\right).$$

Note, that we can also write

$$\mathcal{W}^{\mathcal{N}_A}\,\exp(\mathcal{N}_A) = \exp\left(\mathcal{N}_A(1+log(\mathcal{W}))\right) \leq \exp\left(\mathcal{N}_A + log(m_A)\right).$$

□

Lemma 16.8 will be the basis of our subsequent estimates.

16.6 Bounds for the evolution of the support of the function R^i

To proceed, let us consider the increase the support of the function R^i with respect to T_1 and T_8 in subsequent iterations. Let us denote

$$Supp_i(T_1, T_8) := \cup_{x \in \overline{\Omega}} Supp_x R^i, \tag{16.34}$$

where

$$Supp_x R^i := \{(T_1, T_8); (x, T_1, T_8) \in Supp R^i\}$$

and $Supp R^i$ is the support of the function R^i in the space $\overline{\Omega} \times \mathbb{R}^2$. The following lemma holds.

Lemma 16.9. *Suppose that, given $i \geq 1$, for all $x \in \overline{\Omega}$ we have $R^{i-1}(x, T_1, T_8) = 0$ for $T_1 > T_1^{i-1}$ and $T_8 > T_8^{i-1}$. Then $R^i(x, T_1, T_8) \equiv 0$ for $0 > T_1 > T_1^i = T_1^{i-1} + 2\Delta t$ and $0 > T_8 > T_8^i = T_8^{i-1} + \Delta t$.*

Proof By means of Lemma 16.7, for $\Delta t > 0$ sufficiently small, the expression inside the square bracket multiplying R^i in Eq.(16.5) is positive in $\overline{\Omega}$, so by Lemma 15.1, we conclude that $R^i(x, T_1, T_8) \equiv 0$, if

$$R^{i-1}(x, \tau^{i-1}(x, T_1), \tau_8^{i-1}(x, T_8)) = 0$$

for all $x \in \overline{\Omega}$. By (16.20), we have

$$\tau_1^{i-1}(x, T_1) = T_1 - \gamma(x, T_1)\Delta t > T_1 - 2\Delta t$$

so

$$\tau_1^{i-1}(x, T_1) > T_1^{i-1} \quad \text{for } T_1 > T_1^{i-1} + 2\Delta t.$$

Likewise,

$$\tau_8^{i-1}(x, T_8) > T_8^{i-1} \quad \text{for } T_8 > T_8^{i-1} + \Delta t.$$

The lemma is proved. $\square$

16.7 Estimates of the a priori bounds of the functions $c_1^{u;i}$ and $c_8^{u;i}$

For each $x \in \overline{\Omega}$, we thus have, according to (1.14) and Lemma 16.9

$$\begin{gathered}\int_0^\infty \int_0^\infty c_{8*}^{8;i-1} T_8 \, R^{i-1} \, dT_1 \, dT_8 \leq \|R^{i-1}\| \iint_{Supp_{i-1}} T_8 \, dT_1 \, dT_8 \leq \\ \leq \|R^{i-1}\| \, (T_{1*0}^0 + 2(i-1)\Delta t) \cdot (T_{8*}^0 + (i-1)\Delta t)^2/2 =: \\ \frac{\|R^0\|}{(1 - A\Delta t)^{i-1}} \, W_1(T_{1*}^0, T_{8*}^0, (i-1)\Delta t) =: K_1^{i-1}\end{gathered} \tag{16.35}$$

and

$$\begin{gathered}\int_0^\infty \int_0^\infty c_{1*}^{i-1} T_1 \, R^{i-1} \, dT_1 \, dT_8 \leq \|R^{i-1}\| \iint_{Supp_{i-1}} T_1 \, dT_1 \, dT_8 \leq \\ \leq \|R^{i-1}\| (T_{1*}^0 + 2(i-1)\Delta t)^2/2 \cdot (T_{8*}^0 + (i-1)\Delta t) =: \\ \frac{\|R^0\|}{(1 - A\Delta t)^{i-1}} \, W_8(T_{1*}^0, T_{8*}^0, (i-1)\Delta t) =: K_8^{i-1}\end{gathered} \tag{16.36}$$

where $\|\cdot\| = \|\cdot\|_{L^\infty}$. In (16.35) and (16.36), in accordance with the definition (16.34)

$$T_{1*}^0 := \sup_{Supp_0(T_1, T_8)} T_1, \quad T_{8*}^0 := \sup_{Supp_0(T_1, T_8)} T_8. \tag{16.37}$$

Using the inequalities (16.35) and (16.36), we will find a bound for absolute values of the functions $c_1^{u;i}$ and $c_8^{u;i}$ on the interval $[(i-1)\Delta t, i\Delta t]$ for all $i \in \{1,\ldots,n\}$. These estimates take into account Lemma 16.5 and the inequalities preceding this lemma, where we assumed the non-negativity of the functions $c_1^{u;i}$ and $c_8^{u;i}$. The non-negativity property is inherited by the functions with index $i \geq 1$ from the functions with index $i-1$ via the relation $Supp_{i-1}(T_1, T_8) \subset \overline{\mathbb{R}_+^2}$ (see (16.34)) implied by Lemma 16.5. In view of this, let us note that, given non-negative $c_1^{u;i-1}$ and $c_8^{u;i-1}$, the equations for $c_1^{u;i}$ and $c_8^{u;i}$ can be written as

$$\frac{dc_1^{u;i}}{dt} = \nabla^2 c_1^{u;i} + \widetilde{\nu}\, \mathcal{C}_1^i(x) - c_1^{u;i} \tag{16.38}$$

with the function $\mathcal{C}_1^i(x) \geq 0$ given. The function $c_1^{u;i}$ determined for $(t,x) \in [(i-1)\Delta t, i\Delta t] \times \overline{\Omega}$, satisfies homogeneous Neumann boundary conditions and initial condition $c_1^{u;i}((i-1)\Delta t, x) = c_1^{u;i-1}((i-1)\Delta t, x) \geq 0$. Thus, according to (16.35) and the theory of sub- and supersolutions, the function $c_1^{u;i} \equiv 0$ is a subsolution, whereas a supersolution to Eq. (16.38) on the interval $[(i-1)\Delta t, i\Delta t]$ can be chosen as the solution to the ordinary differential equation of the form

$$\frac{dc_1^{u;i}}{dt} = \widetilde{\nu} K_1^{i-1} - c_1^{u;i} \tag{16.39}$$

where K_1^{i-1} is defined in (16.35). Let us note that the solution to the equation

$$\frac{d}{dt}c = -\omega_1 c + \omega_2 K, \quad c(t_0) = c_0 > 0.$$

equals

$$c(t) = \frac{\omega_2}{\omega_1}\left(1 - e^{-\omega_1(t-t_0)}\right) K + e^{-\omega_1(t-t_0)} c_0.$$

It follows that for $\omega_1 > 0$, $\omega_2 > 0$, $c_0 > 0$ and $t \in [t_0, t_0 + \Delta t)$ we have

$$c(t) \leq \frac{\omega_2}{\omega_1}\left(1 - e^{-\omega_1 \Delta t}\right) K + e^{-\omega_1(t-t_0)} c_0 < \omega_2 \Delta t K + e^{-\omega_1(t-t_0)} c(t_0)$$

and

$$c(t_0 + \Delta t) < \omega_2 \Delta t K + e^{-\omega_1 \Delta t} c(t_0).$$

It thus follows that, for $i = 1, \ldots, n+1$,

$$\|c_1^{u;i}\| < e^{-\Delta t}\|c_1^{u;i-1}\| + \Delta t\ \tilde{\nu} W_1(T_{1*}^0, T_{8*}^0, (i-1)\Delta t)\frac{\|R^0\|}{(1 - A\Delta t)^{i-1}} \tag{16.40}$$

By putting consecutively the estimate for $\|c_1^{u;j-1}\|$ into the estimate for $\|c_1^{u;j}\|$, starting from $j = 1$ up till $j = i$, we obtain

$$\begin{aligned}
\|c_1^{u;i}\| &< e^{-i\Delta t}\|c_1^{u;0}\| + \sum_{j=1}^{i} \widetilde{\nu}\Delta t\ W_1(T_{1*}^0, T_{8*}^0, (j-1)\Delta t)\frac{\|R^0\|}{(1 - A\Delta t)^{j-1}} \leq \\
& e^{-i\Delta t}\|c_1^{u;0}\| + \ \widetilde{\nu} W_1(T_{1*}^0, T_{8*}^0, (i-1)\Delta t)\frac{\|R^0\|}{(1 - A\Delta t)^{i-1}}\sum_{j=1}^{i}\Delta t \leq \\
& e^{-i\Delta t}\|c_1^{u;0}\| + i\Delta t\ \widetilde{\nu} W_1(T_{1*}^0, T_{8*}^0, i\Delta t)\frac{\|R^0\|}{(1 - A\Delta t)^{i-1}}\ .
\end{aligned}$$

Let us note that for all $\Delta t > 0$ sufficiently small (so, due to (16.13), for all n sufficiently large), we have

$$\frac{1}{(1 - A\Delta t)^{n+1}} < \frac{3}{2} e^{AT}$$

hence, by means of arguments leading to (16.32), we arrive at the inequality

$$\|c_1^{u;i}\| < e^{-i\Delta t}\|c_1^{u;0}\| + \frac{3}{2}\, i\Delta t\ \widetilde{\nu} W_1(T_{1*}^0, T_{8*}^0, i\Delta t)\|R^0\| e^{A\, i\Delta t}, \tag{16.41}$$

where $W_1(T_{1*}^0, T_{8*}^0, i\Delta t)$ is defined in (16.35). Likewise, we have

$$\|c_8^{u;i}\| < e^{-\pi_8 i\Delta t}\|c_1^{u;0}\| + \frac{3}{2}\, i\Delta t\ \frac{\widetilde{\mu}}{\pi_8} W_8(T_{1*}^0, T_{8*}^0, i\Delta t)\|R^0\| e^{A\, i\Delta t} \tag{16.42}$$

Using the estimates (16.32),(16.41) and (16.42), we can proceed to further characterize the properties of the functions R^i, $c_1^{u;i}$ and $c_8^{u;i}$. In particular, we can estimate the L^∞ norm of their first derivatives.

Lemma 16.10. *Suppose that for $(t,x) \in (0,T] \times \Omega$, and $a > 0$, u satisfy the equation*

$$\frac{\partial u}{\partial t} = \Delta u - au + f(t,x)$$

$$\frac{\partial u}{\partial \mathbf{n}}(t,x) = 0 \ \ \textit{for}\ x \in \partial\Omega, \quad u(0,x) = \phi(x)$$

and that the compatibility conditions are satisfied, i.e. $\dfrac{\partial \phi}{\partial \mathbf{n}} = 0$ on $\partial\Omega$. Then, for all $\beta \in (0,1)$, the following estimate holds:

$$\|u\|_{C^{(1+\beta)/2,1+\beta}_{t,x}((0,T)\times\Omega)} \le C_p \left[\|f\|_{L^\infty((0,T)\times\Omega)} + \|\phi\|_{C^{1+\beta}_x(\Omega)}\right] \tag{16.43}$$

with the constant C_p depending on β, T and the parameters characterizing Ω.

Proof The estimate (16.48) is a particular version of Theorem 6.49 of section VI in [25]. (Note that $f \in L^\infty(\Omega \times (0,T))$ belongs also to the Morrey space $M_{1,1+m+\beta}$.) □

Remark Let us comment on the membership of the function $f \in L^\infty(\Omega \times (0,T))$ in the Morrey space $M_{1,1+m+\beta}$. According to the definition given before Theorem 7.37 in [25], the Morrey space $M^{p,q}$, $p \in (1,\infty)$, $q \le 2$, can be defined as the subset of the space L^p with the finite norm of the form

$$\|u\|_{p,q} = \sup_{Q(r), r<diam\Omega_T} \left(r^{-q} \int_{Q(r)} \int |u|^p dX \right),$$

where $\Omega_T = [0,T] \times \Omega$ and

$$\|X\| := \max(|x|, |t|^{1/2})$$

with

$$|x| := \left(\sqrt{\sum_{j=1}^m x_j^2} \right).$$

Next (see, sec. I.3 in [25])

$$Q(X_0, r) = \{|x - x_0| < r, |t - t_0| < r^2; t < t_0\}.$$

It follows that as $r \to 0$, then $\int_{Q(r)} dX = O(r^{m+2})$. As $f \in L^\infty(\Omega_T)$, then, for all $\beta \in (0,1)$, $\|f\|_{1,1+m+\beta} < \infty$. □

In applying Lemma 16.10 to Eqs (16.6) and (16.7), let us note that ϕ can be identified with $c_1^{u;0}$ and $c_8^{u;0}$, whereas $f : (0,T] \times \Omega$ can be identified with the functions:

$$\widetilde{\nu} \textstyle\sum_{i=1}^{n+1} \chi_i \int_0^\infty \int_0^\infty c_{8*}^{8;i-1} T_8\, R^{i-1}\, dT_1\, dT_8 = \widetilde{\nu} \sum_{i=1}^{n+1} \chi_i \int_0^\infty \int_0^\infty c_8^{8;i-1}((i-1)\Delta t, x) T_8\, R^{i-1}\, dT_1\, dT_8$$

$$\widetilde{\mu} \textstyle\sum_{i=1}^{n+1} \chi_i \int_0^\infty \int_0^\infty c_{1*}^{i-1} T_1\, R^{i-1}\, dT_1\, dT_8 = \widetilde{\mu} \sum_{i=1}^{n+1} \chi_i \int_0^\infty \int_0^\infty c_1^{i-1}((i-1)\Delta t, x) T_1\, R^{i-1}\, dT_1\, dT_8 \tag{16.44}$$

where χ_i is the characteristic function of the interval $[(i-1)\Delta t, i\Delta t]$. As the integrands in the above integrals are continuous with respect to x on each of the intervals $[(i-1)\Delta t, i\Delta t]$, then these integrals are of $L^\infty([0,T]\times\overline{\Omega})$ class. Let us denote:

$$c_1^u(t,x) := \sum_{i=1}^{n} \chi_i c_1^{u;i}(t,x), \quad c_8^u(t,x) := \sum_{i=1}^{n} \chi_i c_8^{u;i}(t,x) \tag{16.45}$$

where χ_i is the characteristic function of the interval $[(i-1)\Delta t, i\Delta t]$.

Lemma 16.11. *Let $n \geq 3$ be fixed. Suppose that for each $i \in \{0,1,\dots,n\}$ and each $x \in \overline{\Omega}$, the $C^0(\overline{\Omega})$ norms of the functions R^i are bounded from above uniformly with respect to i. Then, for each $\beta \in (0,1)$, c_1^u and c_8^u are of class $C_{t,x}^{(1+\beta)/2,1+\beta}((0,T)\times\Omega)$. To be more precise, there exist constants $C_1(\beta,\Omega)$, $C_8(\beta,\Omega)$, K_1 and K_8, depending on T, such that*

$$\|c_1^u\|_{C_{t,x}^{(1+\beta)/2,1+\beta}((0,T)\times\Omega)} \leq C_1(\beta,\Omega)\left[K_1 + \|c_1^{u;0}\|_{C_x^{1+\beta}(\Omega)}\right] \tag{16.46}$$

and

$$\|c_8^u\|_{C_{t,x}^{(1+\beta)/2,1+\beta}((0,T)\times\Omega)} \leq C_8(\beta,\Omega)\left[K_8 + \|c_8^{u;0}\|_{C_x^{1+\beta}(\Omega)}\right]. \tag{16.47}$$

Proof The lemma follows from Lemma 16.10, together with (16.41) and (16.41). The constants K_1 and K_8 can be chosen as independent on n. □

Lemma 16.12. *Let $n \geq 3$ be fixed. Suppose that for each $i \in \{0,1,\dots,n\}$ the $C^1(\Omega)$ norms of the functions R^i are bounded from above uniformly with respect to i. Then, for each $\beta \in (0,1)$, c_1^u and c_8^u are of class $C_{t,x}^{1+\beta/2,2+\beta}(((i-1)\Delta t, i\Delta t)\times\Omega)$. To be more precise, there exist constants $C_1(\beta,\Omega)$, $C_8(\beta,\Omega)$, K_1 and K_8, depending on T, such that*

$$\|c_1^u\|_{C_{t,x}^{1+\beta/2,2+\beta}(((i-1)\Delta t,i\Delta t)\times\Omega)} \leq C_{1\Delta}(\beta,\Omega)\left[K_1 + \|c_1^u((i-1)\Delta t,\cdot)\|_{C_x^{2+\beta}(\Omega)}\right] \tag{16.48}$$

and

$$\|c_8^u\|_{C_{t,x}^{1+\beta/2,2+\beta}(((i-1)\Delta t,i\Delta t)\times\Omega)} \leq C_{8\Delta}(\beta,\Omega)\left[K_8 + \|c_8^u((i-1)\Delta t,\cdot)\|_{C_x^{2+\beta}(\Omega)}\right]. \tag{16.49}$$

In particular, there exists a constant P independent of i such that as $\Delta t \to 0$

$$\|c_1^u(i\Delta t,\cdot) - c_1((i-1)\Delta t,\cdot)\|_{C^0(\Omega)} \leq P\Delta t, \quad \|c_8^u(i\Delta t,\cdot) - c_8((i-1)\Delta t,\cdot)\|_{C^0(\Omega)} \leq P\Delta t. \tag{16.50}$$

and for all $t \in [(i-1)\Delta t, i\Delta t]$:

$$\|c_1^u(t,\cdot)-c_1((i-1)\Delta t,\cdot)\|_{C^0(\Omega)} \leq P\Big(t-(i-1)\Delta t\Big), \quad \|c_8^u(t,\cdot)-c_8((i-1)\Delta t,\cdot)\|_{C^0(\Omega)} \leq P\Big(t-(i-1)\Delta t\Big). \tag{16.51}$$

Proof The lemma follows from Lemma 16.10, according to which c_1^u and c_8^u (defined in (16.45)) are of $C_{t,x}^{(1+\beta)/2,1+\beta}((0,T)\times\Omega)$ class. Starting from the initial data equal to $c_1^{u,0}$ and $c_8^{u,0}$ (and assuming that they are of $C_x^{2+\beta}(\Omega)$ class) we obtain a $C_{t,x}^{1+\beta/2,2+\beta}$ solution on the set $([0,\Delta t)\times\Omega)$. Treating $c_1^{u;1}(t=\Delta t,x)$ and $c_8^{u;1}(t=\Delta t,x)$ as the initial data on the interval we obtain a solution of $C_{t,x}^{1+\beta/2,2+\beta}$ class on the set $([1\cdot\Delta t, 2\cdot\Delta t)\times\Omega)$. Proceeding consecutively in this way, we obtain a $C_{t,x}^{1+\beta/2,2+\beta}$ solution on the set $([(i-1)\cdot\Delta t, i\cdot\Delta t)\times\Omega)$ for all $i \in \{1,\dots,n\}$, hence using the Schauder esimates, we obtain inequalities (16.48) and (16.49). As the constants K_1 and K_8 can be chosen as independent on n and i, then, due to the fact that the time derivative of the solutions is bounded (and Holder continuous), there exists a constant P such that for $\Delta t > 0$ sufficiently small, inequality (16.50) holds. □

Remark The following counterpart of inequalities (16.48) and (16.49) follow straightforwardly from Lemma 16.10:

$$\|c_1^u\|_{C_{t,x}^{(1+\beta)/2,1+\beta}(((i-1)\Delta t,i\Delta t)\times\Omega)} \le C_{1\Delta}(\beta,\Omega)\left[K_{1\Delta i} + \|c_1^u((i-1)\Delta t,\cdot)\|_{C_x^{1+\beta}(\Omega)}\right] \tag{16.52}$$

and

$$\|c_8^u\|_{C_{t,x}^{(1+\beta)/2,1+\beta}(((i-1)\Delta t,i\Delta t)\times\Omega)} \le C_{8\Delta}(\beta,\Omega)\left[K_{8\Delta i} + \|c_8^u((i-1)\Delta t,\cdot)\|_{C_x^{1+\beta}(\Omega)}\right]. \tag{16.53}$$

□

16.8 Estimates of first order derivatives of R^i with respect to T_m

In this section, we will examine the differentiability properties of solutions to system (16.5)-(16.7) (together with (16.8)-(16.9)) with respect to the variables T_1 and T_8. Differentiating Eq.(16.5) with respect to T_1 we obtain for any pair (T_1, T_8):

$$\begin{aligned}0 = d_R\nabla^2 B_1^i - B_1^i&\left[\frac{\partial}{\partial T_1}\left(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\right) + \frac{\partial}{\partial T_8}\left(\delta(c_{8*}^{u;i-1}, T_8)\right) + F_1((i-1)\Delta t, x, T_1, T_8) + \frac{1}{\Delta t}\right]\\ -R^i&\Big[\frac{\partial^2}{\partial T_1^2}\left(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\right) + \frac{\partial}{\partial T_1}F_1((i-1)\Delta t, x, T_1, T_8)\Big]\\ +F_0&((i-1)\Delta t, x)\cdot\nabla B_1^i + \frac{B_1^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})}{\Delta t},\end{aligned} \tag{16.54}$$

where

$$B_1^i(x,T_1,T_8) := \frac{\partial R^i}{\partial T_1}(x,T_1,T_8).$$

Recall that, according to (16.8) and (16.9),

$$\tau_1^{i-1} = T_1 - \Delta t\cdot\left(\gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)\right), \tag{16.55}$$

$$\tau_8^{i-1} = T_8 - \Delta t\cdot\delta(c_{8*}^{u;i-1}(x), T_8). \tag{16.56}$$

It follows that

$$\begin{aligned}\frac{\partial\tau_1^{i-1}}{\partial T_1} &= 1 - \Delta t\,\frac{\partial\gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial T_1},\\ \frac{\partial\tau_8^{i-1}}{\partial T_8} &= 1 - \Delta t\,\frac{\partial\delta(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_8)}{\partial T_8},\end{aligned} \tag{16.57}$$

hence

$$\begin{aligned}&B_1^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1}) := \frac{\partial R^{i-1}}{\partial\tau_1^{i-1}}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\frac{\partial\tau_1^{i-1}}{\partial T_1} =\\ &\frac{\partial R^{i-1}}{\partial\tau_1^{i-1}}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\left(1 - \Delta t\,\frac{\partial\gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial T_1}\right)\end{aligned} \tag{16.58}$$

and

$$\begin{aligned}&B_8^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1}) := \frac{\partial R^{i-1}}{\partial\tau_8^{i-1}}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\frac{\partial\tau_8^{i-1}}{\partial T_8} =\\ &\frac{\partial R^{i-1}}{\partial\tau_8^{i-1}}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\left(1 - \Delta t\,\frac{\partial\delta(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_8)}{\partial T_8}\right).\end{aligned} \tag{16.59}$$

Now, let us fix (T_1, T_8) and, for given $x \in \overline{\Omega}$, $(\tau_1^{i-1}, \tau_8^{i-1})$ as well. In this way, we can treat $B_1^i(\cdot, T_1, T_8)$ and $B_1^{i-1}(\cdot; \tau_1^{i-1}(\cdot, T_1), \tau_8^{i-1}(\cdot, T_8))$ as functions of x only. If extremum of the absolute value of $B_1^i(x, T_1, T_8)$ is attained at $y \in \Omega$, then

$$\nabla B_1^i(y, T_1, T_8) = 0,$$

and, due to the maximum principle, (16.58) and Lemma 16.7:

$$\begin{aligned}
&|B_1^i(y, T_1, T_8)| \leq \Big(R^i(y, T_1, T_8) \Big| \frac{\partial^2}{\partial T_1^2} \gamma(c_{1*}^{u;i-1}(y), c_{8*}^{u;i-1}(y), T_1) + \\
&\frac{\partial}{\partial T_1} F_1((i-1)\Delta t, y, T_1, T_8) \Big| + \Big| \frac{B_1^{i-1}(y; \tau_1^{i-1}, \tau_8^{i-1})}{\Delta t} \Big| \Big) \left(-A + \frac{1}{\Delta t} \right)^{-1} \leq \\
&\left(R^i(y, T_1, T_8)(A_M + f_1) + \Big| \frac{B_1^{i-1}(y; \tau_1^{i-1}, \tau_8^{i-1})}{\Delta t} \Big| \right) \left(-A + \frac{1}{\Delta t} \right)^{-1} = \\
&\Big(R^i(y, T_1, T_8)(A_M + f_1)\Delta t + |B_{\tau_1}^{i-1}(y; \tau_1^{i-1}, \tau_8^{i-1})|(1 + A\Delta t) \Big) \, (1 - A\Delta t)^{-1} ,
\end{aligned} \tag{16.60}$$

where

$$A_M := \max\{A_-, A_+\} \tag{16.61}$$

with A_-, A_+ defined in inequality (16.23) of Lemma 16.7, and A defined by (16.30).

Now, let us suppose that the global extremum of the absolute value of $B_1^i(x, T_1, T_8)$ is attained at $y \in \partial\Omega$. Suppose that this extremum is a positive maximum. Then, from the fact that $\partial R^i(x, T_1, T_8)/\partial n(x) = 0$, we conclude that

$$\frac{\partial B_1^i(x, T_1, T_8)}{\partial n(x)} = 0 \quad \text{for } x \in \partial\Omega, \tag{16.62}$$

hence

$$\frac{\partial^2 B_1^i(y, T_1, T_8)}{\partial n(y)^2} \leq 0$$

as otherwise $B_1^i(x, T_1, T_8)$ would not have attained a maximum at $x = y \in \partial\Omega$. Next, the Laplacian of $B_1^i(x, T_1, T_8)$ for $x \in \partial\Omega$ is equal to the sum of the second derivatives with respect to $n(x)$ and the second derivatives with respect to the directions lying in the plane tangent to $\partial\Omega$ at x. This is seen from the form of the Laplace operator with respect to variables locally connected with $\partial\Omega$ supplied by the Appendix A. Due to the fact that B_1^i has a maximum at $x = y$, each of these derivatives is non-positive. It follows that $\nabla^2 B_1^i(y, T_1, T_8) \leq 0$, hence the estimate of the form (16.60) holds. It should be noted that, according to Assumption 16.2, in this case the terms proportional to F_0 and F_1 do not take part in the estimates, because they are identically equal to zero at $\partial\Omega$. The same arguments can be applied in the case, when the extremum is a non-positive minimum.

As we showed above, thanks to the assumption concerning the compactness of the initial data, for each $i \in \{1, \dots, n\}$ the points (T_1, T_8) for which $R^i(x, T_1, T_8) \not\equiv 0$ are contained in a compact set $\mathcal{S}^i$. For all $i \in \{1, \dots, n\}$, $y = y(T_1, T_8)$, so we can take a supremum over $(T_1, T_8) \in \mathcal{S}^i$. In this way, we obtain the estimate for

$$\mathcal{B}_1^i := \sup_{(T_1, T_8) \in \mathcal{S}^i} |B_1^i(y(T_1, T_8), T_1, T_8)|.$$

$\mathcal{B}_1^0$ is given by the initial conditions. Next, let us note that, if

$$L := (1 - A\Delta t)^{-1} \tag{16.63}$$

then

$$(1 + A\Delta t) < L, \tag{16.64}$$

hence, for $i \geq 1$ we have,

$$\mathcal{B}_1^i \leq \left(\|R^i\|A_f(\Delta t) + \mathcal{B}_1^{i-1}L\right)L\,, \tag{16.65}$$

where

$$A_f = A_M + f_1. \tag{16.66}$$

In arriving to (16.88) we used the fact that

$$\sup_{x\in\Omega,\tau_1,\tau_8} |B_{\tau_1}^{i-1}(x,\tau_1,\tau_8)| \leq \sup_{x\in\Omega,T_1,T_8} |B_1^{i-1}(x,T_1,T_8)| \tag{16.67}$$

We have

$$\mathcal{B}_1^1 \leq \left(\|R^1\|A_f(\Delta t) + \mathcal{B}_1^0 L\right)L,$$

$$\mathcal{B}_1^2 \leq \left(\|R^2\|A_f(\Delta t) + \mathcal{B}_1^1 L\right)L \leq \left(\|R^2\|A_f(\Delta t) + (\|R^1\|A_f(\Delta t) + \mathcal{B}_1^0 L)LL\right)L$$

so inductively, for $i \in \{3,\ldots,n(\Delta t)\}$,

$$\mathcal{B}_1^i \leq \mathcal{B}_1^0 L^{2i} + A_f(\Delta t)\sum_{j=1}^{i}\|R^j\|L^{2(i-j)+1}.$$

Using (16.31) and (16.63), we obtain

$$\mathcal{B}_1^i \leq L^{2i}\left(\mathcal{B}_1^0 + A_f(\Delta t)\textstyle\sum_{j=1}^{i}\|R^j\|L^{-2j+1}\right) \leq L^{2i}\left(\mathcal{B}_1^0 + A_f(\Delta t)\|R^0\|\textstyle\sum_{j=1}^{i}L^{-j+1}\right) \leq$$

$$L^{2i}\left(\mathcal{B}_1^0 + A_f(\Delta t)\|R^0\|\textstyle\sum_{j=1}^{i}L^{-j+1}\right) \leq L^{2i}\left(\mathcal{B}_1^0 + A_f(\Delta t)i\|R^0\|\right).$$

so consequently, as $i\Delta t = \dfrac{i}{n}T$, we have by means of Remark after (16.32), for $\Delta t > 0$ sufficiently small:

$$\mathcal{B}_1^i \leq \frac{3}{2}A_f\, i\Delta t\|R^0\|_{L^\infty}\exp(2A\frac{i}{n}T) + \frac{3}{2}\mathcal{B}_1^0\exp(2A\frac{i}{n}T). \tag{16.68}$$

Denoting $t := i\Delta t$, we can write:

$$\mathcal{B}_1^i \leq \frac{3}{2}A_f t\|R^0\|_{L^\infty}\exp(2At) + \frac{3}{2}\mathcal{B}_1^0\exp(2At) \tag{16.69}$$

Likewise, we have the estimate

$$\mathcal{B}_8^i \leq \frac{3}{2}A_f\, i\Delta t\|R^0\|_{L^\infty}\exp(2A\frac{i}{n}T) + \frac{3}{2}\mathcal{B}_8^0\exp(2A\frac{i}{n}T). \tag{16.70}$$

which, after inserting $t := i\Delta t$, can be written as:

$$\mathcal{B}_8^i \leq \frac{3}{2}A_f t\|R^0\|_{L^\infty}\exp(2At) + \frac{3}{2}\mathcal{B}_8^0\exp(2At), \tag{16.71}$$

where

$$\mathcal{B}_8^i := \sup_{(T_1,T_8)\in\mathcal{S}^i} |B_8^i(y(T_1,T_8),T_1,T_8)|.$$

with

$$B_8^i(x,T_1,T_8) := \frac{\partial R^i}{\partial T_8}(x,T_1,T_8).$$

16.9 Estimates of the first order derivatives of R^i with respect to the components of x inside Ω

We will start from the estimates of the absolute values of the first derivatives of the functions R^i with respect to the components of x attained inside Ω.

Remark To avoid confusion, the derivative of $R^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))$ with respect to τ_k^{i-1}, $k=1,8$, will be denoted below by $B_{\tau_k}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))$, where, for simplicity, we have omitted the index $i-1$. □

Differentiating Eq.(16.5) with respect to x_r, $r=1,2,3$, we obtain the equation:

$$\begin{aligned}
&0 = d_R\nabla^2 Q_r^i - Q_r^i\left[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big)+\frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1},T_8)\Big)+F_1^{i-1}+\frac{1}{\Delta t}\right]-\\
&R^i\Big[\frac{\partial^2}{\partial c_{1*}^{u;i-1}\partial T_1}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big)c_{1*,x_r}^{u;i-1}+\frac{\partial^2}{\partial c_{8*}^{u;i-1}\partial T_1}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big)c_{8*,x_r}^{u;i-1}+\\
&\frac{\partial^2}{\partial c_{8*}^{u;i-1}\partial T_8}\Big(\delta(c_{8*}^{u;i-1},T_8)\Big)c_{8*,x_r}^{u;i-1}+\frac{\partial F_1^{i-1}}{\partial x_r}\Big]+\frac{Q^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})}{\Delta t}+\\
&\frac{B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))}{\Delta t}\cdot\frac{\partial\tau_1^{i-1}}{\partial x_r}+\frac{B_{\tau_8}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))}{\Delta t}\cdot\frac{\partial\tau_8^{i-1}}{\partial x_r}+\\
&F_0^{i-1}\cdot\nabla\left(\frac{\partial R^i}{\partial x_r}\right)+\frac{\partial F_0^{i-1}}{\partial x_r}\cdot\nabla R^i,
\end{aligned}\tag{16.72}$$

where, for $i\in\{1,\dots,n\}$,

$$Q_r^i(x,T_1,T_8) := \frac{\partial R^i}{\partial x_r}(x,T_1,T_8)$$

and, for simplicity we denoted

$$F_0^{i-1} := F_0((i-1)\Delta t,x),\quad F_1^{i-1} := F_1((i-1)\Delta t,x,T_1,T_8).$$

According to (16.8) we have:

$$\begin{aligned}
&\frac{1}{\Delta t}\cdot\frac{\partial\tau_1^{i-1}}{\partial x_l} = -\frac{\partial\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)}{\partial x_l} =\\
&-\frac{\partial\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)}{\partial c_{1*}^{u;i-1}}\cdot\frac{\partial c_{1*}^{u;i-1}}{\partial x_l}-\frac{\partial\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)}{\partial c_{8*}^{u;i-1}}\cdot\frac{\partial c_{1*}^{u;i-1}}{\partial x_l}
\end{aligned}\tag{16.73}$$

and, according to (16.9) :

$$\frac{1}{\Delta t}\cdot\frac{\partial\tau_8^{i-1}}{\partial x_l} = -\frac{\partial\delta(c_{8*}^{u;i-1}(x),T_8)}{\partial x_l} = -\frac{\partial\widetilde{\delta}(c_{8*}^{u;i-1},T_8)}{\partial c_{1*}^{u;i-1}}\cdot\frac{\partial c_{8*}^{u;i-1}}{\partial x_l}\tag{16.74}$$

hence (16.72) can be written as

$$
\begin{aligned}
0 = d_R \nabla^2 Q_r^i - Q_r^i \left[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1^{i-1} + \frac{1}{\Delta t}\right] \\
+ \frac{Q_r^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})}{\Delta t} + \Big\{ - R^i \Big[\frac{\partial^2}{\partial c_{1*}^{u;i-1}\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big)\, c_{1*,x_r}^{u;i-1} \\
+ \frac{\partial^2}{\partial c_{8*}^{u;i-1}\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big)\, c_{8*,x_r}^{u;i-1} + \frac{\partial^2}{\partial c_{8*}^{u;i-1}\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big)\, c_{8*,x_r}^{u;i-1} + \frac{\partial F_1^{i-1}}{\partial x_r}\Big] \\
- B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) \cdot \frac{\partial \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial x_r} - \\
B_{\tau_8}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) \cdot \frac{\partial \delta(c_{8*}^{u;i-1}(x), T_8)}{\partial x_r}\Big\} + F_0^{i-1}\cdot\nabla\left(\frac{\partial R^i}{\partial x_r}\right) + \frac{\partial F_0^{i-1}}{\partial x_r}\cdot \nabla R^i\,.
\end{aligned}
\tag{16.75}
$$

Now, using (16.26) and Lemma 16.11, we conclude that

$$
\Big|\frac{\partial^2}{\partial c_{1*}^{u;i-1}\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big)\, c_{1*,x_r}^{u;i-1} + \frac{\partial^2}{\partial c_{8*}^{u;i-1}\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big)\, c_{8*,x_r}^{u;i-1}\Big| \leq
\mathcal{M}\left(C_1(\beta,\Omega)\|c_1^{u;0}\|_{C_x^{1+\beta}(\Omega)} + C_8(\beta,\Omega)\|c_8^{u;0}\|_{C_x^{1+\beta}(\Omega)}\right).
$$

Recall that the constants $C_8(\beta,\Omega)$, and $C_8(\beta,\Omega)$ are independent of i, as, in fact, the x-derivatives of the functions defined by (16.45) are Hölder continuous in t for $t \in [0,T]$.

Next, using (16.25) and Lemma 16.11, we conclude that:

$$
\Big|\frac{\partial \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial x_r}\Big| \leq T_{1i}\,\mathcal{M}\left(C_1(\beta,\Omega)\|c_1^{u;0}\|_{C_x^{1+\beta}(\Omega)} + C_8(\beta,\Omega)\|c_8^{u;0}\|_{C_x^{1+\beta}(\Omega)}\right), \tag{16.76}
$$

where T_{1i} is estimated from above in Lemma 16.9 as $T_{1*}^0 + 2i\Delta t$.

The corresponding inequalities for the function δ take the form:

$$
\Big|\frac{\partial^2}{\partial c_{8*}^{u;i-1}\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big)\, c_{8*,x_r}^{u;i-1}\Big| \leq \delta_2\, C_8(\beta,\Omega)\|c_8^{u;0}\|_{C_x^{1+\beta}(\Omega)}
$$

(see (16.24)) and

$$
\Big|\frac{\partial \delta(c_{8*}^{u;i-1}(x), T_8)}{\partial x_r}\Big| \leq T_{8i}\delta_2 C_8(\beta,\Omega)\|c_8^{u;0}\|_{C_x^{1+\beta}(\Omega)}, \tag{16.77}
$$

where T_{8i} is estimated from above in Lemma 16.9 as $T_{8*}^0 + i\Delta t$. Let

$$
G_1 := \mathcal{M}\Big(C_1(\beta,\Omega)\|c_1^{u;0}\|_{C_x^{1+\beta}(\Omega)} + C_8(\beta,\Omega)\|c_8^{u;0}\|_{C_x^{1+\beta}(\Omega)}\Big)\Big|_{\beta=0},\; G_8 := \Big(\delta_2\, C_8(\beta,\Omega)\|c_8^{u;0}\|_{C_x^{1+\beta}(\Omega)}\Big)\Big|_{\beta=0}. \tag{16.78}
$$

Let

$$
\sup_{r=1,2,3} |Q_r^i| = |Q_\rho^i|
$$

and

$$
Q^i := \Big|Q_\rho^i\Big|.
$$

for some $\rho \in \{1,2,3\}$. Obviously Q^i (as well as Q_r^i) are functions of T_1 and T_8.

Now, the absolute value of the term $F_0^{i-1}\cdot\nabla\left(\frac{\partial R^i}{\partial x_\rho}\right) + \frac{\partial F_0^{i-1}}{\partial x_\rho}\cdot\nabla R^i$ at a point of the global extremum, where $\nabla \dfrac{\partial R^i}{\partial x_\rho} = \nabla Q^i = 0$ can be estimated as

$$\Big|\frac{\partial F_0^{i-1}}{\partial x_\rho} \cdot \nabla R^i\Big| \le 3 f_0 Q^i. \tag{16.79}$$

In this way we can find an L^∞ estimate for the expression in the curly bracket in (16.75) (including the term $\partial F_1^{i-1}/\partial x_r$ with its absolute value estimated from above by f_1) as:

$$\Big\|\Big\{\cdot\Big\}\Big\|_{L^\infty} \le \|R^i\|_{L^\infty} \cdot (G_1 + G_8 + f_1) + \mathcal{B}_1^i \cdot G_1 \cdot (T_{1*}^0 + 2n\Delta t) + \mathcal{B}_8^i \cdot G_8 \cdot (T_{8*}^0 + n\Delta t), \tag{16.80}$$

where G_1 and G_8 are defined in (16.78). Using inequalities (16.68), (16.70) we can write

$$\Big\|\Big\{\cdot\Big\}\Big\|_{L^\infty} \le S_Q \exp(2A\frac{i}{n}T),$$

where, for $n = T/\Delta t$,

$$S_Q := \frac{3}{2}\|R^0\|_{L^\infty} \cdot (G_1 + G_8 + f_1) + b_1\, G_1 \cdot (T_{1*}^0 + 2T) + b_8\, G_8 \cdot (T_{8*} + T)$$

with

$$b_1 = \frac{3}{2} A_f\, T \|R^0\|_{L^\infty} + \frac{3}{2}\mathcal{B}_1^0$$

and

$$b_8 = \frac{3}{2} A_f\, T \|R^0\|_{L^\infty} + \frac{3}{2}\mathcal{B}_8^0.$$

Lemma 16.13. *Let $T > 0$, $s \in \mathbb{N}$ and $\mathbb{N} \ni n \gg 1$ be fixed. Let $\Delta t = Tn^{-1}$ and $L := (1 - A\Delta t)^{-1}$, $A > 0$. Then, for all n sufficiently large, we have:*

$$\sigma_n = \Delta t \left(1 + L^s + L^{2s} + \ldots + L^{ns}\right) < \frac{3}{2}\frac{1}{sA}\exp(Ans\Delta t).$$

Next, for any $i \le n$,

$$\sigma_i := \Delta t \left(1 + L^s + L^{2s} + \ldots + L^{is}\right) \le \frac{3}{2}\frac{1}{A}(\exp(Ans\Delta t))^{i/n} = \frac{3}{2}\frac{1}{sA}\exp(Ais\Delta t).$$

Proof By (16.64), we have $L^s > (1 + A\Delta t)^s > 1 + sA\Delta t$, hence

$$\sigma_n = \Delta t \left(1 + L^s + L^{2s} + \ldots, L^{ns}\right) = \Delta t \frac{L^{(n+1)s} - 1}{L^s - 1} < \Delta t \frac{L^{(n+1)s}}{L^s - 1} < \frac{L^{(n+1)s}}{sA}.$$

Next, for n sufficiently large, we have

$$L^{(n+1)s} = \Big(\frac{1}{1 - A\dfrac{T}{n}}\Big)^{ns}\Big(\frac{1}{1 - A\dfrac{T}{n}}\Big)^{s} = \Big(\frac{1}{1 - sA\dfrac{T}{ns}}\Big)^{ns}\Big(\frac{1}{1 - A\dfrac{T}{n}}\Big)^{s} < \frac{3}{2}\exp(AsT)$$

thus, for $\Delta t \to 0$,

$$\sigma_n < \frac{3}{2}\frac{1}{A}\exp(Asn\Delta t).$$

In general, for $i \le n$, we have

$$\sigma_i < \frac{3}{2}\frac{1}{sA}\exp(A\frac{i}{n}sT) = \frac{3}{2}\frac{1}{sA}\exp(Ais\Delta t).$$

The lemma is proved. □

Now, recall that by Lemma 16.7 the first two terms in the bracket multiplying Q_r^i in (16.75) are uniformly bounded from below for all non-negative values of their arguments by the constant $(-A)$.

In consequence, for fixed $t \in [0,T]$ and n satisfying (16.13), it follows from (16.75) by means of Lemma 15.1, that, for $i = 1, \dots, n$, the estimate from above of

$$\mathcal{Q}^i := \sup_{T_1,T_8} Q^i = \sup_{x\in\Omega, T_1, T_8, r=1,2,3} \Big| \frac{\partial R^i}{\partial x_r}(x, T_1, T_8) \Big|, \tag{16.81}$$

in relation to the value of $\mathcal{Q}^{i-1}$, can be written as

$$\mathcal{Q}^i \leq \frac{\mathcal{Q}^{i-1} + S_Q \Delta t \, \exp(2A\frac{i}{n}T)}{1 - (A + 3f_0)\Delta t} < \frac{\mathcal{Q}^{i-1} + S_Q \Delta t \, \exp(2AT)}{(1 - A_1)}$$

where A is defined in (16.30),

$$A_1 = (A + 3f_0) \tag{16.82}$$

and $\mathcal{Q}^{i-1}$ corresponds to $Q^{i-1}(x; \tau_1^{i-1}, \tau_8^{i-1})$. Denoting, similarly as in (16.63), $L := (1 - A_1 \Delta t)^{-1}$, we thus have

$$\mathcal{Q}^1 \leq \Big(\mathcal{Q}^0 + S_Q \Delta t \exp(2AT) \Big) L,$$

$$\mathcal{Q}^2 \leq \left(\mathcal{Q}^1 + S_Q \Delta t \exp(2AT) \right) L = \left(\left(\mathcal{Q}^0 + S_Q \Delta t \exp(2AT) \right) L + S_Q \Delta t \exp(2AT) \right) L =$$

$$\mathcal{Q}^0 L^2 + S_Q \Delta t \exp(2AT) \left(L^2 + L^1 \right)$$

and by induction, for any $i \leq n$,

$$\mathcal{Q}^i \leq \mathcal{Q}^0 L^i + S_Q \Delta t \exp(2AT) \left(L^i + \ldots + L^2 + L^1 \right).$$

Using Lemma 16.13 and proceeding as in the proof of (16.68) and (16.69), we can show that for n sufficiently large (and $\Delta t > 0$ satisfying equality (16.13)), for all $i \in \{1, \dots, n\}$, we have:

$$\mathcal{Q}^i \leq \frac{3}{2} \frac{1}{A_1} S_Q \exp(2AT + A_1 \frac{i}{n} T) + \frac{3}{2} \mathcal{Q}^0 \exp(A_1 \frac{i}{n} T). \tag{16.83}$$

Thus the C^1_x norm of the function R^i can be estimated by the C^1_x norm of the initial conditions $\mathcal{Q}^0$, where $\mathcal{Q}^0$ is defined in (16.81).

16.10 Estimates of the first order derivatives of R^i with respect to the components of x at $\partial\Omega$

Let us consider the case when an extremum of the absolute value of the spatial derivative is attained on the boundary $\partial\Omega$. Suppose that in the initial system of coordinates,

$$M_1 := \max_j \sup_{x\in\overline{\Omega}} \Big\{ \Big| \frac{\partial R^i}{\partial x_j}(x) \Big| \Big\} = \Big| \frac{\partial R^i}{\partial x_r}(x_0) \Big|$$

for some $r \in \{1, 2, 3\}$ and $x_0 \in \partial\Omega$. Without losing generality, we can assume that this extremum of the absolute value corresponds to a positive maximum of $R^i_{,x_r}$.

If $\hat{x}_r \parallel n(x_0)$, then $\dfrac{\partial R^i}{\partial x_r} = 0$. So, suppose that $\hat{x}_r \nparallel n(x_0)$. Let $N(x_0) := \hat{x}_r$. We can decompose

$$N(x_0) = \frac{1}{\sqrt{3}} \left(n(x_0) + s_1(x_0) + s_2(x_0) \right),$$

where $n(x_0)$ is a unit vector outward-normal to the boundary $\partial\Omega$ at $x = x_0$ and unit vectors $s_1(x_0)$, $s(x_0)$ belong to a space tangent to $\partial\Omega$ at $x = x_0$. By appropriate rotations of the system of coordinates, we can achieve that $n(x_0) = \hat{x}_3$ and $s_l(x_0) = \hat{x}_l$ for $l = 1, 2$. In the (possibly) new system of coordinates, we have

$$N(x_0)\cdot(\nabla R^i)(x_0)=\frac{1}{\sqrt{3}}\left(n(x_0)+s_1(x_0)+s_2(x_0)\right)\cdot(\nabla R^i)(x_0)=$$

$$\frac{1}{\sqrt{3}}\left(\hat{x}_3+\hat{x}_1+\hat{x}_2\right)\left(\hat{x}_3\frac{\partial R^i}{\partial x_3}+\hat{x}_1\frac{\partial R^i}{\partial x_1}+\hat{x}_2\frac{\partial R^i}{\partial x_2}\right)=\frac{1}{\sqrt{3}}\left(\frac{\partial R^i}{\partial x_1}+\frac{\partial R^i}{\partial x_2}\right).$$

It follows that (after appropriate rotation of coordinate system around the axis parallel to $\hat{x}_3$) it suffices to consider the derivative $\dfrac{\partial R^i}{\partial x_l}$ with $\hat{x}_l \perp n(x_0)$ and $l=1,2$. Then

$$\frac{\partial}{\partial x_3}\left(\frac{\partial R^i}{\partial x_l}\right)=\frac{\partial}{\partial x_l}\left(\frac{\partial R^i}{\partial x_3}\right)=0.$$

where from we obtain

$$\frac{\partial}{\partial x_3}\left(\frac{\partial R^i}{\partial x_l}\right)=0. \tag{16.84}$$

As we assumed that $R^i_{,x_l}>0$ at $x=x_0$, then

$$\frac{\partial^2 R^i_{,x_l}}{\partial x_3^2}\le 0$$

at $x=x_0$. Next, as it was assumed that $R^i_{,x_l}$ has a positive maximum as $x=x_0$, then $\dfrac{\partial R^i_{,x_l}}{\partial x_1}=\dfrac{\partial R^i_{,x_l}}{\partial x_2}=0$ and the second order derivatives of $R^i_{,x_l}$ with respect to x_1 and x_2 are non-positive. Using (16.84) and the lemma from Appendix A (with S identified with $\partial\Omega$ in the vicinity of x_0), we infer that at $x=x_0$ we have $\Delta R^i_{,x_l}\le 0$ and to estimate the value of $\dfrac{\partial R^i}{\partial x_r}(x_0)$, we can use the maximum principle as if $x_0\in\Omega$. The same reasoning holds if the extremum is a non-positive minimum, i.e. $\dfrac{\partial R^i}{\partial x_r}(x_0)\le 0$.

16.11 Second order derivatives of R^i with respect to T_l and T_m

We will show how to estimate the second derivative $\partial^2 R^i/\partial T_1^2$. The other second order derivatives with respect to T_1 and T_8 variables can be estimated in the similar way. Differentiating the both sides of the equation (16.54) with respect to T_1, we obtain

$$\begin{aligned}
0=\;& d_R\nabla^2 B^i_{11}-B^i_{11}\left[\frac{\partial}{\partial T_1}\left(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\right)+\frac{\partial}{\partial T_8}\left(\delta(c_{8*}^{u;i-1},T_8)\right)+F_1((i-1)\Delta t,x,T_1,T_8)+\frac{1}{\Delta t}\right]\\
&-2B^i_1\left[\frac{\partial^2}{\partial T_1^2}\left(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\right)+\frac{\partial F_1}{\partial T_1}((i-1)\Delta t,x,T_1,T_8)\right]\\
&-R^i\Big[\frac{\partial^3}{\partial T_1^3}\left(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\right)+\frac{\partial^2 F_1}{\partial T_1^2}((i-1)\Delta t,x,T_1,T_8)\Big]\\
&+\frac{B^{i-1}_{11}(x;\tau^i_1,\tau^i_8)}{\Delta t}+\;F_0((i-1)\Delta t,x)\cdot\nabla B^i_{11},
\end{aligned} \tag{16.85}$$

where

$$B^i_1(x,T_1,T_8):=\frac{\partial R^i}{\partial T_1}(x,T_1,T_8)\quad\text{and}\quad B^i_{11}(x,T_1,T_8):=\frac{\partial^2 R^i}{\partial T_1^2}(x,T_1,T_8).$$

The term $B^{i-1}_{11}(x,\tau^{i-1}_1,\tau^{i-1}_8)$ is defined as:

$$
\begin{aligned}
&B_{11}^{i-1}(x,\tau_1^{i-1},\tau_8^{i-1}) = B_{11}^{i-1}(x,\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) = \\
&\frac{d^2R^{i-1}}{dT_1^2}(x,\tau_1^{i-1},\tau_8^{i-1}) = \frac{d}{dT_1}\left(\frac{\partial R^{i-1}}{\partial \tau_1}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\left(1-\Delta t\,\frac{\partial\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)}{\partial T_1}\right)\right) = \\
&\frac{\partial^2 R^{i-1}}{\partial \tau_1^2}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\left(1-\Delta t\,\frac{\partial\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)}{\partial T_1}\right)^2 - \\
&\Delta t\;\frac{\partial R^{i-1}}{\partial \tau_1}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\;\frac{\partial^2\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)}{\partial T_1^2} = \\
&B_{\tau_1\tau_1}^{i-1}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\left(1-\Delta t\,\frac{\partial\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)}{\partial T_1}\right)^2 - \\
&\Delta t\;B_{\tau_1}^{i-1}(x,\tau_1^{i-1},\tau_8^{i-1})\cdot\;\frac{\partial^2\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)}{\partial T_1^2}\,.
\end{aligned}
\tag{16.86}
$$

Assuming that

$$
\mathcal{B}_{11}^i := \sup_{x\in\Omega,T_1,T_8}|B_{11}^i(x,T_1,T_8)|
$$

is attained for $y\in\Omega$ and some (T_1,T_8), we conclude that

$$
\nabla B_{11}^i(y,T_1,T_8) = 0.
$$

Thus taking into account (16.67), the inequality

$$
\sup_{x\in\Omega,\tau_1,\tau_8}|B_{\tau_1\tau_1}^{i-1}(x,\tau_1,\tau_8)| \le \sup_{x\in\Omega,T_1,T_8}|B_{11}^{i-1}(x,T_1,T_8)|, \tag{16.87}
$$

using the maximum principle and proceeding as in arriving at (16.60), we obtain the relation

$$
\begin{aligned}
&\mathcal{B}_{11}^i \le \left(\|R^i\|A_{31} + 2\mathcal{B}_1^i(A_- + f_1) + \mathcal{B}_1^{i-1}A_M + \frac{\mathcal{B}_{11}^{i-1}(1+\Delta t A)^2}{\Delta t}\right)\left(-A+\frac{1}{\Delta t}\right)^{-1} = \\
&\Big(\|R^i\|A_{31}\Delta t + 2\mathcal{B}_1^i(A_- + f_1)\Delta t + \mathcal{B}_1^{i-1}A_M\Delta t + \mathcal{B}_{11}^{i-1}(1+A\Delta t)^2\Big)\,(1-A\Delta t)^{-1} \le \\
&\Big(\Big[\|R^i\|A_{31} + 2\mathcal{B}_1^iA_f + \mathcal{B}_1^{i-1}A_M\Big]\Delta t + \mathcal{B}_{11}^{i-1}L^2\Big)\,L
\end{aligned}
\tag{16.88}
$$

where A is defined in (16.30), L is defined by (16.63), A_M in (16.61), A_f defined in (16.66), whereas

$$
A_{31} = \sup\left|\frac{\partial^3}{\partial T_1^3}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big)\right| + f_1. \tag{16.89}
$$

According to (16.68), we can write

$$
\begin{aligned}
&\|R^i\|A_{31} + 2\mathcal{B}_1^iA_f + \mathcal{B}_1^{i-1}A_M \le \frac{3}{2}\|R^0\|\cdot(A_{31}+2A_fA_fi\Delta t)\,\exp(2A\frac{i}{n}T) + 3A_f\mathcal{B}_1^0\exp(2A\frac{i}{n}T) + \\
&\frac{3}{2}\|R^0\|\,A_fA_M(i-1)\Delta t\exp(2A\frac{i-1}{n}T) + \frac{3}{2}A_M\mathcal{B}_1^0\exp(2A\frac{i-1}{n}T) \le \\
&\exp(2AT)\Big\{(\frac{3}{2}A_{31}+3A_f^2T)\|R^0\| + 3A\mathcal{B}_1^0 + \frac{3}{2}\|R^0\|A_fA_MT + \frac{3}{2}A_M\mathcal{B}_1^0\Big\} =: S_{11}(T).
\end{aligned}
\tag{16.90}
$$

We have $\mathcal{B}_{11}^1 \le \Big(S_{11}(T)\Delta t + \mathcal{B}_{11}^0L^2\Big)L$, $\mathcal{B}_{11}^2 \le \Big(S_{11}(T)\Delta t + S_{11}L^3\Delta t + \mathcal{B}_{11}^0L^5\Big)L$ and in general, for $i\le n$,

$$\mathcal{B}_{11}^i = \Big(S_{11}(T)\Delta t \sum_{l=0}^{i-1} L^{3l} + \mathcal{B}_{11}^0 L^{3i-1}\Big) L$$

Thus using Lemma 16.13, we obtain, for $\Delta t > 0$ sufficiently small, i.e. for L sufficiently close to 1

$$\mathcal{B}_{11}^i \leq \frac{3}{2}\frac{1}{3A} S_{11}(T)\exp(3iA\Delta t) + \frac{3}{2}\mathcal{B}_{11}^0 \exp(3iA\Delta t). \tag{16.91}$$

The estimate for $\mathcal{B}_{88}^i$ has a simpler form due to the fact that $\partial^2\delta/\partial T_8^2 \equiv 0$. This implies that the last term in the expression for B_{88}^{i-1} (corresponding to (16.86)), hence the term corresponding to $\mathcal{B}_1^{i-1} A_M$ in (16.88), equals zero. Next, A_{31} in (16.88) reduces to f_1. Similarly to $\mathcal{B}_{11}^i$, we have

$$\mathcal{B}_{88}^i \leq \frac{3}{2}\frac{1}{3A} S_{88}(T)\exp(3iA\Delta t) + \frac{3}{2}\mathcal{B}_{88}^0 \exp(3iA\Delta t), \tag{16.92}$$

but this time, according to (16.90),

$$S_{88}(T) = \frac{3}{2}\,\|R^0\| f_1\, \exp(2AT) + 2\mathcal{B}_8^n A_f,$$

where $\mathcal{B}_8^n$ is obtained from (16.70) by taking $i = n$.

Similar estimates can be found for $\mathcal{B}_{18}^i$. In this case the second line of (16.85) takes the form

$$B_8^i \left[\frac{\partial^2}{\partial T_1^2}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial F_1}{\partial T_1}((i-1)\Delta t, x, T_1, T_8)\right] + B_1^i \left[\frac{\partial F_1}{\partial T_8}((i-1)\Delta t, x, T_1, T_8)\right],$$

whereas the third one takes the form

$$R^i \Big[\frac{\partial^2 F_1}{\partial T_1 T_8}((i-1)\Delta t, x, T_1, T_8)\Big]$$

We thus have

$$\mathcal{B}_{18}^i \leq \frac{3}{2}\frac{1}{3A} S_{18}(T)\exp(3iA\Delta t) + \frac{3}{2}\mathcal{B}_{18}^0 \exp(3iA\Delta t), \tag{16.93}$$

with

$$S_{18}(T) = \frac{3}{2}\,\|R^0\| f_1\, \exp(2AT) + (\mathcal{B}_8^n + \mathcal{B}_1^n)\, A_f,$$

where $\mathcal{B}_1^n$ and $\mathcal{B}_8^n$ are obtained from (16.68) and (16.70) by taking $i = n$.

16.12 Third order derivatives of R^i with respect to T_l, T_m and T_p

Using the same approach we are able to give estimates for third order derivatives of the functions with respect to T_l, T_m and T_p, $l, m, p \in \{1, 8\}$, which are independent of i. Similarly to (16.91), (16.92) and (16.93), these estimates have the following structure:

$$\mathcal{B}_{lmp}^i \leq \mathcal{S}_{lmp}\exp(k_{3;1} iA\Delta t) + \mathcal{B}_{lmp}^0 \exp(k_{3;2} iA\Delta t), \tag{16.94}$$

where $k_{3;1}$ and $k_{3;2}$ are finite natural numbers and S_{lmp} depends on T and the norms of the coefficient functions of system (16.5)-(16.7).

16.13 Mixed second order derivatives of R^i with respect to x_k and T_m

Now, we will estimate the absolute values of the mixed derivatives

$$Q_{k,m}^i := \frac{\partial^2 R^i}{\partial x_k \partial T_m} = \frac{d^2 R^i}{dx_k dT_m}, \quad k = 1, 2, 3,\ m = 1, 8.$$

Let $g_1 := \gamma$, $g_8 := \delta$. Then

$$\begin{aligned}
&\frac{d}{dx_k}\Big[\frac{\partial R^{i-1}}{\partial T_m}(x,\tau_1^{i-1},\tau_8^{i-1})\Big] = \frac{d}{dT_m}\Big[\frac{dR^{i-1}}{dx_k}(x,\tau_1^{i-1},\tau_8^{i-1})\Big] = \\
&\frac{d}{dT_m}\Big[\frac{\partial R^{i-1}}{\partial x_k}(x,\tau_1^{i-1},\tau_8^{i-1}) + B_{\tau_1}^{i-1}(x,\tau_1^{i-1},\tau_8^{i-1})\frac{\partial \tau_1^{i-1}}{\partial x_k} + B_{\tau_8}^{i-1}(x,\tau_1^{i-1},\tau_8^{i-1})\frac{\partial \tau_8^{i-1}}{\partial x_k}\Big] = \\
&\frac{d}{dT_m}\Big[B_{\tau_1}^{i-1}(x,\tau_1^{i-1},\tau_8^{i-1})\frac{\partial \tau_1^{i-1}}{\partial x_k} + B_{\tau_8}^{i-1}(x,\tau_1^{i-1},\tau_8^{i-1})\frac{\partial \tau_8^{i-1}}{\partial x_k}\Big] + \\
&Q_{k,\tau_m}^{i-1}(x,\tau_1^{i-1},\tau_8^{i-1}) \cdot \left(1 - \Delta t\,\frac{\partial g_m(c_1^{u;i-1}(x),c_8^{u;i-1}(x),T_m)}{\partial T_m}\right).
\end{aligned} \tag{16.95}$$

Differentiating Eq.(16.75) with respect to T_m, we obtain the equation:

$$\begin{aligned}
&0 = d_R\nabla^2 Q_{k,m}^i - Q_{k,m}^i\left[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1},T_8)\Big)\right. \\
&\left.+F_1((i-1)\Delta t,T_1,T_8) + \frac{1}{\Delta t}\right] - Q_k^i\Big[\frac{\partial^2\gamma}{\partial^2 T_1}\delta_{1m} + \frac{\partial F_1^i}{\partial T_m}((i-1)\Delta t,x,T_1,T_8)\Big] \\
&+\frac{Q_{k,\tau_m}^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})}{\Delta t}\left(1-\Delta t\,\frac{\partial g_m(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_m)}{\partial T_m}\right) \\
&+\Big\{-B_m^i\Big[\frac{\partial^2}{\partial c_{1*}^u\partial T_1}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big)c_{1*,x_k}^{u;i-1} + \frac{\partial^2}{\partial c_{8*}^u\partial T_1}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big)c_{8*,x_k}^{u;i-1} + \\
&\frac{\partial^2}{\partial c_{8*}^u\partial T_8}\Big(\delta(c_{8*}^{u;i-1},T_8)\Big)c_{8*,x_k}^{u;i-1} + \frac{\partial F_1^{i-1}}{\partial x_k}((i-1)\Delta t,x,T_1,T_8)\Big] - \\
&R^i\Big[\frac{\partial^3}{\partial c_{1*}^u\partial^2 T_m}\Big(g_m(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_m)\Big)c_{1*,x_k}^{u;i-1} + \frac{\partial^3}{\partial c_{8*}^u\partial^2 T_m}\Big(g_m(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_m)\Big)c_{8*,x_k}^{u;i-1} + \\
&\frac{\partial^2 F_1^{i-1}}{\partial x_k\partial T_m}((i-1)\Delta t,x,T_1,T_8)\Big] + \\
&B_{\tau_1\tau_m}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))\;\frac{1}{\Delta t}\cdot\frac{\partial\tau_1^{i-1}}{\partial x_k}\cdot\frac{\partial\tau_m}{\partial T_m} + \\
&B_{\tau_8\tau_m}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))\;\frac{1}{\Delta t}\cdot\frac{\partial\tau_8^{i-1}}{\partial x_k}\cdot\frac{\partial\tau_m}{\partial T_m} + \\
&B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))\;\frac{1}{\Delta t}\cdot\frac{\partial}{\partial T_m}\left(\frac{\partial\tau_1^{i-1}}{\partial x_k}\right)\delta_{1m} + \\
&B_{\tau_8}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))\;\frac{1}{\Delta t}\cdot\frac{\partial}{\partial T_m}\left(\frac{\partial\tau_8^{i-1}}{\partial x_k}\right)\delta_{8m}\;\Big\} + \\
&F_0^{i-1}(x)\cdot\nabla Q_{k,m(x,T_1,T_8)} + \frac{\partial F_0^{i-1}}{\partial x_k}\cdot\nabla B_m(x,T_1,T_8).
\end{aligned} \tag{16.96}$$

Recall that $\partial\tau_m/\partial T_m$ has the form determined by (16.57), whereas $\partial\tau_l/\partial x_k$ is determined in (16.73) and (16.74). It follows from (16.73) and (16.74) that

$$\begin{aligned}
&\frac{1}{\Delta t}\cdot\frac{\partial}{\partial T_m}\left(\frac{\partial\tau_j^{i-1}}{\partial x_l}\right) = -\frac{\partial}{\partial T_m}\left(\frac{\partial g_j(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_j)}{\partial x_l}\right)\delta_{jm} = \\
&-\frac{\partial^2 g_m(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_m)}{\partial c_{1*}^{u;i-1}\partial T_m}\cdot\frac{\partial c_{1*}^{u;i-1}}{\partial x_l} - \frac{\partial^2 g_m(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)}{\partial c_{8*}^{u;i-1}}\cdot\frac{\partial c_{1*}^{u;i-1}}{\partial x_l}
\end{aligned} \tag{16.97}$$

hence, by Lemma 16.7, (16.76) and (16.77), the L^∞-norm of the function coefficients multiplying $B_{\tau_1\tau_m}^{i-1}$, $B_{\tau_8\tau_m}^{i-1}$, $B_{\tau_1}^{i-1}$ and $B_{\tau_8}^{i-1}$ in (16.96) are bounded by a finite number $C_{xT_*}^B$ depending on (T_1,T_8) (and other parameters of the system). Likewise, the coefficient function multiplying R^i is bounded in its L^∞-norm by a finite number $C^R_{xT_*}$. Recall also that the sum of the absolute values of the coefficients multiplying B^i_m can be estimated from above by a finite number (Lemma 16.11).

Let us estimate the maximal absolute value of the derivatives $Q^i_{k,m}$ **inside** Ω, for fixed $(T_1, T_8) \in \overline{\mathbb{R}}^2_+$. Suppose that the maximum of $|Q^i_{k,m}(\cdot, T_1, T_8)|$ is realized at some point

$$x^i_{k,m} = x^i_{k,m}(T_1, T_8) \in \Omega$$

thus

$$\nabla Q^i_{k,m}(x^i_{k,m}, T_1, T_8) = 0.$$

Let

$$\sup_{k=1,2,3;\, m=1,8;T_1,T_8} |Q^i_{k,m}(x_{km}, T_1, T_8)| = |Q^i_{\bar{k},\bar{m}}(x_{\bar{k}\bar{m}}, \bar{T}_1, \bar{T}_8)| =: |Q^i_{\bar{k},\bar{m}}(\bar{x}, \bar{T}_1, \bar{T}_8)| =: \mathcal{Q}^i_{*,*} \tag{16.98}$$

Obviously, $(\nabla_x Q^i_{\bar{k},\bar{m}})(\bar{x}, \bar{T}_1, \bar{T}_8) = 0$ and, according to the definition of $Q^i_{\bar{k},\bar{m}}(\bar{x}, \bar{T}_1, \bar{T}_8)$, we have

$$\Big|\frac{\partial F^i_0}{\partial x_k} \cdot \nabla B_{\bar{m}}(\bar{x}, \bar{T}_1, \bar{T}_8)\Big| \leq 3 f_0 |Q^i_{\bar{k},\bar{m}}(\bar{x}, \bar{T}_1, \bar{T}_8)|. \tag{16.99}$$

Next, proceeding like in sections 16.9 and 16.11, and using the estimates derived there, we can show that the following recurrence inequality holds:

$$Q^i_{*,*} \leq \Big(E_{*,*} \exp[5AT]\Delta t + Q^{i-1}_{*,*} L\Big) L,$$

where $E_{*,*}$ depends on the initial data R_0, T and the coefficient functions of system (16.5)-(16.7). This leads to the inequality

$$\mathcal{Q}^i_{*,*} \leq \frac{3}{2} E_{*,*} \exp[5AT] \exp(2Ai\Delta t) + \frac{3}{2} \mathcal{Q}^0_{*,*} \exp(2Ai\Delta t). \tag{16.100}$$

Remark Let us note that we calculated the maximal value of $|Q^i_{k,m}|$ tacitly assuming that it is attained inside Ω. In fact, the same procedure can be applied in the case when $|Q^i_{k,m}|$ attains it's maximal value at the boundary $\partial\Omega$. This follows from subsection 16.10 and the fact that the boundary properties of $Q^i_{k,m}$ are the same as the properties of Q^i_k. □

Remark The necessity of taking the supremum over the index k (and m) as in (16.98) follows from the presence of the term $F_0 \cdot \nabla R^i$ and is dictated by the possibility of the assessment (16.99). Without this term, we could keep the indices k and m fixed. □

16.14 Mixed third order derivatives R^i with respect to x_k, T_m and T_l

Similarly, we can estimate the absolute value third order derivatives of the form:

$$Q^i_{k,m,l} := \frac{\partial^3 R^i}{\partial x_k \partial T_m \partial T_l} = \frac{d^3 R^i}{dx_k dT_m dT_l}, \quad k = 1, 2, 3,\ m = 1, 8.$$

These estimates have the form corresponding to (16.101). Thus, in view of the second Remark after (16.101), we have:

$$\mathcal{Q}^i_{*,**} \leq \frac{3}{2} E_{*,**} \exp(k_{12;1} Ai\Delta t) + \frac{3}{2} \mathcal{Q}^0_{*,**} \exp(k_{12;2} Ai\Delta t), \tag{16.101}$$

where for the first asterisk we can take $1, 2, 3$, whereas for the second and third asterisk we can take 1 or 8. The constants $k_{1,2;1}$, $k_{1,2;2}$ are finite natural numbers. $E_{*,**}$ depend on T and the norms of the coefficient functions of system (16.5)-(16.7).

16.15 Basic lemma concerning the difference between functions corresponding to subsequent values of i

For further analysis, we will impose a simplifying technical assumption.

Assumption 16.14. *The function F_0 does not depend on t.*

In this subsection, using the results of the previous subsections, we will estimate the difference

$$Z^i(x, T_1, T_8) := R^i(x, T_1, T_8) - R^{i-1}(x, T_1, T_8).$$

For $i \geq 2$, Z^i satisfies the equation:

$$d_R \nabla^2 Z^i + F_0(x) \cdot \nabla Z^i - \frac{Z^i(x; T_1, T_8)}{\Delta t} + \frac{Z^{i-1}(x, T_1, T_8)}{\Delta t} - \frac{Z_*^{i-1}(x, T_1, T_8)}{\Delta t} -$$
$$\left\{ Z^i \left[\frac{\partial}{\partial T_1} \left(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) \right) + \frac{\partial}{\partial T_8} \left(\delta(c_{8*}^{u;i-1}, T_8) \right) + F_1((i-1)\Delta t, x, T_1, T_8) \right] \right\} + R^{i-1} \Delta V^i = 0 \tag{16.102}$$

where

$$Z_*^{i-1} := (R_*^{i-1} - R^{i-1}) - (R_*^{i-2} - R^{i-2}), \tag{16.103}$$

because

$$R^i - R_*^{i-1} - (R^{i-1} - R_*^{i-2}) = (R^i - R^{i-1}) - (R^{i-1} - R^{i-2}) + (R^{i-1} - R_*^{i-1}) - (R^{i-2} - R_*^{i-2}).$$

Above, for fixed (T_1, T_8) we denoted for brevity

$$R_*^k := R^k(x, \tau_1(x, T_1), \tau_8(x, T_8)) \tag{16.104}$$

with $\tau_1(x, T_1), \tau_8(x, T_8)$ determined by (16.8)-(16.9), i.e.

$$\tau_1^{i-1} = T_1 - \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1) \cdot \Delta t =: T_1 - \gamma^{i-1}(x, T_1) \cdot \Delta t, \tag{16.105}$$

$$\tau_8^{i-1} = T_8 - \delta(c_{8*}^{u;i-1}(x), T_8) \cdot \Delta t =: T_8 - \delta^{i-1}(x, T_8) \cdot \Delta t \tag{16.106}$$

and

$$\begin{aligned} \Delta V^i = &\left[\frac{\partial}{\partial T_1} \left(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) \right) + \frac{\partial}{\partial T_8} \left(\delta(c_{8*}^{u;i-1}, T_8) \right) + F_1((i-1)\Delta t, x, T_1, T_8) \right] - \\ &\left[\frac{\partial}{\partial T_1} \left(\gamma(c_{1*}^{u;i-2}, c_{8*}^{u;i-2}, T_1) \right) + \frac{\partial}{\partial T_8} \left(\delta(c_{8*}^{u;i-2}, T_8) \right) + F_1((i-2)\Delta t, x, T_1, T_8) \right]. \end{aligned} \tag{16.107}$$

In view of the inequality (16.50),

$$\Delta V^i \leq B_v \Delta t \quad \text{as} \quad \Delta t \to 0$$

for all $i \geq \{2, \ldots, n\}$ and some $B_v \geq 0$.

Remark Above, we used the following lemma specifying the one term Taylor expansion for many variables scalar function.

Lemma 16.15. *Suppose that $f \in C^{\mathcal{K}+1}$ class. Then*

$$f(\mathbf{y}) = f(\mathbf{y}_0) + \sum_{1 \leq |\alpha| \leq \mathcal{K}} \frac{1}{\alpha!} (D^\alpha f)(\mathbf{y}_0)(\mathbf{y} - \mathbf{y}_0)^\alpha + \sum_{|\alpha| = \mathcal{K}+1} \frac{\mathcal{K}+1}{\alpha!} (\mathbf{y} - \mathbf{y}_0)^\alpha \int_0^1 (1-s)^{\mathcal{K}} (D^\alpha f)(\mathbf{y}_0 + s(\mathbf{y} - \mathbf{y}_0)) \, ds.$$

□

Taking into account the boundedness of the first and second second derivatives of the functions R^i, $i \in \{1, \ldots, n\}$ with respect to T_1 and T_8 provided by sections 16.8, 16.11, we can write:

$$R^{i-1} - R_*^{i-1} = R^{i-1}(x, T_1, T_8) - R^{i-1}(x, \tau_1(x, T_1), \tau_8(x, T_8)) =$$

$$R^{i-1}(x, T_1, T_8) - R^{i-1}(x, T_1, T_8) - B_1^{i-1}(x, T_1, T_8) \cdot \gamma^{i-1}(x, T_1)\Delta t - B_8^{i-1}(x, T_1, T_8) \cdot \delta^{i-1}(x, T_8)\Delta t -$$

$$\sum_{k,l=1,8} \frac{2}{2!}(\tau_k^{i-1} - T_k) \cdot (\tau_l^{i-1} - T_l) \int_0^1 (1-s) B_{\tau_k \tau_l}^{i-1}\Big(x, T_1 + s(\tau_1^{i-1} - T_1), T_8 + s(\tau_8^{i-1} - T_8)\Big)\, ds =$$

$$-B_1^{i-1}(x, T_1, T_8) \cdot \gamma^{i-1}(x, T_1)\Delta t - B_8^{i-1}(x, T_1, T_8) \cdot \delta^{i-1}(x, T_8)\Delta t -$$

$$\sum_{k,l=1,8} (\tau_k^{i-1} - T_k) \cdot (\tau_l^{i-1} - T_l) \int_0^1 (1-s) B_{\tau_k \tau_l}^{i-1}\Big(x, T_1 + s(\tau_1^{i-1} - T_1), T_8 + s(\tau_8^{i-1} - T_8)\Big)\, ds. \tag{16.108}$$

Remark In accordance with Remark before (16.72), to avoid ambiguity, we will assume the following convention of denoting the total derivatives of the quantities $R^i(x, \tau_1^i(x, T_1), \tau_8(x, T_8))$ with respect to T_1 and T_8. Thus, these derivatives will be denoted by

$$\frac{dR^i(x, \tau_1^i(x, T_1), \tau_8(x, T_8))}{dT_1} = B_{\tau_1}^i(x, \tau_1^i(x, T_1), \tau_8(x, T_8)) \cdot \frac{d\tau_1}{dT_1} =: B_1^i(x, \tau_1^i(x, T_1), \tau_8(x, T_8)),$$
$$\frac{dR^i(x, \tau_1^i(x, T_1), \tau_8(x, T_8))}{dT_8} = B_{\tau_8}^i(x, \tau_1^i(x, T_1), \tau_8(x, T_8)) \cdot \frac{d\tau_8}{dT_1} =: B_8^i(x, \tau_1^i(x, T_1), \tau_8(x, T_8)).$$

□

Next

$$B_1^{i-1}(x, T_1, T_8) \cdot \gamma^{i-1}(x, T_1)\Delta t - B_1^{i-2}(x, T_1, T_8) \cdot \gamma^{i-2}(x, T_1)\Delta t =$$

$$\Big(B_1^{i-1}(x, T_1, T_8) - B_1^{i-2}(x, T_1, T_8)\Big) \cdot \gamma^{i-1}(x, T_1)\Delta t + B_1^{i-2}(x, T_1, T_8)\Big(\gamma^{i-1}(x, T_1) - \gamma^{i-2}(x, T_1)\Big)\Delta t :=$$

$$H_1^{i-1}(x, T_1, T_8) \cdot \gamma^{i-1}(x, T_1)\Delta t + B_1^{i-2}(x, T_1, T_8)\Big(\gamma^{i-1}(x, T_1) - \gamma^{i-2}(x, T_1)\Big)\Delta t.$$

Likewise:

$$B_8^{i-1}(x, T_1, T_8) \cdot \delta^{i-1}(x, T_1)\Delta t - B_8^{i-2}(x, T_1, T_8) \cdot \delta^{i-2}(x, T_1)\Delta t =$$

$$\Big(B_8^{i-1}(x, T_1, T_8) - B_8^{i-2}(x, T_1, T_8)\Big) \cdot \delta^{i-1}(x, T_1)\Delta t + B_1^{i-2}(x, T_1, T_8)\Big(\delta^{i-1}(x, T_1) - \delta^{i-2}(x, T_1)\Big)\Delta t :=$$

$$H_8^{i-1}(x, T_1, T_8) \cdot \delta^{i-1}(x, T_1)\Delta t + B_8^{i-2}(x, T_1, T_8)\Big(\delta^{i-1}(x, T_1) - \delta^{i-2}(x, T_1)\Big)\Delta t.$$

As $(\tau_k^{i-1} - T_k) = O(\Delta t)$, it follows that there exists a positive constant r_1 independent of i such that

$$|(R_*^{i-1} - R^{i-1}) - (R_*^{i-2} - R^{i-2})| \leq \Big(|H_1^{i-1}|\overline{\gamma} + |H_8^{i-1}|\overline{\delta}\Big)\Delta t + r_1(\Delta t)^2 \tag{16.109}$$

and positive constants r_2, r_3 (independent of i) such that

$$|Z^i| \leq (|Z^{i-1}| + r_2(\Delta t)^2 + r_3|H|^{i-1}\Delta t)L, \tag{16.110}$$

The last inequality is obtained via the consecutive use of the maximum principle and the denotation

$$|H|^i := \sup\{|H_1^i|, |H_8^i|\}, \tag{16.111}$$

As R^0 is given by the initial data, then

$$\begin{aligned}
&d_R\nabla^2 Z^1 - \frac{Z^1(x;T_1,T_8)}{\Delta t} + F_0(x)\cdot\nabla Z^1 + \\
&\Big\{R^1\left[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;0}, c_{8*}^{u;0}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;0}, T_8)\Big) + F_1(0\cdot\Delta t, x, T_1, T_8)\right] + \\
&\left(\Delta_{xx}R^0 + \frac{[R^0(x,\tau_1^0(x,T_1),\tau_8^0(x,T_8)) - R^0(x,T_1,T_8)]}{\Delta t}\right) + F_0(x)\cdot\nabla R^0\Big\} \;=\; 0.
\end{aligned} \tag{16.112}$$

It is seen that the terms in the curly brackets $\{\cdot\}$ are of $O(1)$ terms as $\Delta t \to 0$. It follows that there exists a constant $G_{0Z}^1 \geq 0$ such that for all $i \in \{2,\ldots,n\}$

$$|Z_1(x)| < \Delta t\, G_{0Z}^1, \quad \text{for } x \in \overline{\Omega}. \tag{16.113}$$

To proceed, let us analyse the difference $H_1^i = B_1^i - B_1^{i-1}$ for $i = \{1,\ldots,n\}$. This will be done by means of the equation obtained by subtracting from Eq.(16.54) for B_1^i the corresponding equation for the function B_1^{i-1}. For $i \geq 2$, we obtain:

$$\begin{aligned}
&d_R\nabla^2 H_1^i + F_0(x)\cdot\nabla H_1^i = \\
&H_1^i\left[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1((i-1)\Delta t, x, T_1, T_8) + \frac{1}{\Delta t}\right] + \\
&B_1^{i-1}[\Delta V_i] + Z^i\Big[\frac{\partial^2}{\partial T_1^2}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_1}F_1((i-1)\Delta t, x, T_1, T_8)\Big] + \\
&R^{i-1}\Big[\frac{\partial^2}{\partial T_1^2}\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) - (\frac{\partial^2}{\partial T_1^2}\gamma(c_{1*}^{u;i-2}, c_{8*}^{u;i-2}, T_1)) + \\
&\frac{\partial}{\partial T_1}F_1((i-1)\Delta t, x, T_1, T_8) - \frac{\partial}{\partial T_1}F_1((i-2)\Delta t, x, T_1, T_8)\Big] - \\
&\frac{B_1^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_1^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))}{\Delta t}.
\end{aligned} \tag{16.114}$$

Denoting

$$\gamma_{T_1}^i := \frac{\partial\gamma(c_1^{u;i}, c_8^{u,8}, T_1)}{\partial T_1},\ \delta_{T_8}^i := \frac{\partial\delta(c_8^{u,8}, T_1)}{\partial T_8},\quad B_{\tau_1}^{i-1} := \frac{\partial R^{i-1}}{\partial\tau_1},\ B_{\tau_1}^{i-2} := \frac{\partial R^{i-2}}{\partial\tau_1},$$

we have:

$$\begin{aligned}
&\Big|B_1^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_1^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))\Big| = \\
&\Big|B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))(1-\gamma_{,T_1}^{i-1}\Delta t) - B_{\tau_1}^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))(1-\gamma_{,T_1}^{i-2}\Delta t)\Big| = \\
&\Big|\ \Big[B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_{\tau_1}^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))\Big]\cdot(1-\gamma_{,T_1}^{i-1}\Delta t) - \\
&B_{\tau_1}^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))\cdot\Big[\gamma_{,T_1}^{i-1}\Delta t - \gamma_{,T_1}^{i-2}\Delta t\Big]\ \Big|
\end{aligned} \tag{16.115}$$

Let us note that

$$|\gamma^{i-1}_{,T_1}(x,T_1)\Delta t - \gamma^{i-2}_{,T_1}(x,T_1)\Delta t| = |\left(\gamma^{i-1}_{,T_1}(x,T_1) - \gamma^{i-2}_{,T_1}(x,T_1)\right)|\Delta t < G_{11}(\Delta t)^2.$$

Next, we have, according to Lemma 16.15, with $\mathcal{K}=0$,

$$\begin{aligned}
&B^{i-1}_{\tau_1}(x;\tau^{i-1}_1(x,T_1),\tau^{i-1}_8(x,T_8)) - B^{i-2}_{\tau_1}(x;\tau^{i-2}_1(x,T_1),\tau^{i-2}_8(x,T_8)) =\\
&B^{i-1}_{\tau_1}(x;\tau^{i-1}_1(x,T_1),\tau^{i-1}_8(x,T_8))\\
&-B^{i-2}_{\tau_1}\Big[x;\tau^{i-1}_1(x,T_1)-\{\tau^{i-1}_1(x,T_1)-\tau^{i-2}_1(x,T_1)\},\tau^{i-1}_8(x,T_8)-\left\{\tau^{i-1}_8(x,T_8)-\tau^{i-2}_8(x,T_8)\right\}\Big] =\\
&B^{i-1}_{\tau_1}(x;\tau^{i-1}_1(x,T_1),\tau^{i-1}_8(x,T_8)) - B^{i-2}_{\tau_1}(x;\tau^{i-1}_1(x,T_1),\tau^{i-1}_8(x,T_8))-\\
&\Big[(\tau^{i-1}_1(x,T_1)-\tau^{i-2}_1(x,T_1))\int_0^1 (B^{i-1}_{\tau_1\tau_1}(x,\tau^{i-1}_1+s(\tau^{i-2}_1-\tau^{i-1}_1),\tau^{i-1}_8+s(\tau^{i-2}_8-\tau^{i-1}_8))\,ds+\\
&(\tau^{i-1}_8(x,T_8)-\tau^{i-2}_8(x,T_8))\int_0^1 (B^{i-2}_{\tau_1\tau_8}(x,\tau^{i-1}_1+s(\tau^{i-2}_1-\tau^{i-1}_1),\tau^{i-1}_8+s(\tau^{i-2}_8-\tau^{i-1}_8))\,ds\Big].
\end{aligned} \tag{16.116}$$

Now, we have the identity:

$$d_R\nabla^2 B^0_1 + F_0(0\cdot\Delta t,x)\cdot\nabla B^0_1 - \frac{B^0_1(x;T_1,T_8)}{\Delta t} - \left\{d_R\nabla^2 B^0_1 + F_0(0\cdot\Delta t,x)\cdot\nabla B^0_1 - \frac{B^0_1(x;T_1,T_8)}{\Delta t}\right\} = 0.$$

By subtracting this identity from (16.54) for $i=1$, and taking into account Assumption 16.14, we conclude that, for $i=1$, (16.114) is substituted by the equation:

$$\begin{aligned}
&0 = d_R\nabla^2 H^1_1 + F_0(x)\cdot\nabla H^1_1 - H^1_1\,\frac{1}{\Delta t} -\\
&B^1_1\left[\frac{\partial}{\partial T_1}\left(\gamma(c^{u;i-1}_{1*},c^{u;i-1}_{8*},T_1)\right) + \frac{\partial}{\partial T_8}\left(\delta(c^{u;i-1}_{8*},T_8)\right) + F_1(0\cdot\Delta t,x,T_1,T_8)\right] -\\
&R^1\Big[\frac{\partial^2}{\partial T_1^2}\left(\gamma(c^{u;i-1}_{1*},c^{u;i-1}_{8*},T_1) + \frac{\partial}{\partial T_1}F_1(0\cdot\Delta t,x,T_1,T_8)\right)\Big]+\\
&F_0(0\cdot\Delta t,x)\cdot\nabla B^0_1 + d_R\nabla^2 B^0_1 + \frac{B^0_1(x;\tau^0_1(x,T_1),\tau^0_8(x,T_8)) - B^0_1(x;T_1,T_8)}{\Delta t}.
\end{aligned} \tag{16.117}$$

Taking into account that

$$\begin{aligned}
&\Big|\frac{B^0_1(x;\tau^0_1(x,T_1),\tau^0_8(x,T_8)) - B^0_1(x;T_1,T_8)}{\Delta t}\Big| =\\
&\left|\frac{B^0_{\tau_1}(x;\tau^0_1(x,T_1),\tau^0_8(x,T_8))\dfrac{\partial\tau^0_1}{\partial T_1} - B^0_1(x;T_1,T_8)}{\Delta t}\right|
\end{aligned}$$

assuming the sufficient smoothness of the initial data and using the maximum principle we obtain the inequality

$$\left|H^1_1\right| < h^1_1\Delta t. \tag{16.118}$$

Likewise, we can obtain the estimate

$$\left|H^1_8\right| < h^1_8\Delta t. \tag{16.119}$$

Using these estimates, we can consider the equation for Z^2, given by (16.102) with $i=2$. Let us note that, due to (16.118) and (16.119), Z^1_* given by (16.103) can be estimated as

$$|Z_*^1| = |(R^1 - R_*^1) - (R^0 - R_*^0)| \leq$$

$$|B_{\tau_1}^1 \tau_1(c_1^{u;0}(x), c_8^{u;0}(x), T_1) \cdot \Delta t - B_{\tau_1}^0 \tau_1(c_1^{u;0}(x), c_8^{u;0}(x), T_1)| + 2\mathcal{B}_{11}(\Delta t)^2 +$$

$$|B_{\tau_8}^1 \tau_8(c_8^{u;0}(x), T_8) \cdot \Delta t - B_{\tau_8}^0 \tau_8(c_8^{u;0}(x), T_8)| + 2\mathcal{B}_8(\Delta t)^2 \leq$$

$$\Big(|h_1^1 \gamma(c_1^{u;0}(x), c_8^{u;0}(x), T_1)| + 2\mathcal{B}_{11}\Big)(\Delta t)^2 + \Big(|h_8^1 \gamma(c_1^{u;0}(x), c_8^{u;0}(x), T_1)| + 2\mathcal{B}_{18}\Big)(\Delta t)^2,$$

hence

$$|Z_*^1| \leq \left(h_1^1 \overline{\gamma} + h_8^1 \overline{\delta}\right)(\Delta t)^2 + (2\mathcal{B}_{11} + 2\mathcal{B}_{88})(\Delta t)^2, \tag{16.120}$$

where

$$\overline{\gamma} = \sup|\gamma|, \quad \overline{\delta} = \sup|\delta|. \tag{16.121}$$

It thus follows from (16.102) and (16.120) by means of the maximum principle that

$$|Z^2| \leq (|Z^1| + K_{1*}(\Delta t)^2)L,$$

for some constant K_{1*}, where $L = (1 - A\Delta t)^{-1}$.

The crucial fact for further analysis is contained in the following lemma.

Lemma 16.16. *Let $x \in \overline{\Omega}$ and (T_1, T_8) be fixed. Then*

$$-(\tau_1^{i-2}(x, T_1) - \tau_1^{i-1}(x, T_1)) = (\gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1) - \gamma(c_{1*}^{u;i-2}(x), c_{8*}^{u;i-2}(x), T_1)) \cdot \Delta t$$

and

$$-(\tau_8^{i-2}(x, T_8) - \tau_8^{i-1}(x, T_8)) = (\delta(c_{8*}^{u;i-1}(x), T_8) - \delta(c_{8*}^{u;i-2}(x), T_8)) \cdot \Delta t$$

hence

$$| - \left(\tau_1^{i-2}(x, T_1) - \tau_1^{i-1}(x, T_1)\right)| \leq G_{21}(\Delta t)^2,$$

where G_{21} is independent of i, and

$$| - \left(\tau_8^{i-2}(x, T_8) - \tau_8^{i-1}(x, T_8)\right)| \leq G_{28}(\Delta t)^2,$$

where G_{28} is independent of i.

Moreover, for $p = 1, 8$,

$$\Big|\frac{\partial \gamma}{\partial c_{p*}^{u;i-1}}(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1) - \frac{\partial \gamma}{\partial c_{p*}^{u;i-2}}(c_{1*}^{u;i-2}(x), c_{8*}^{u;i-2}(x), T_1)\Big| \leq G_{c\gamma p}\Delta t. \tag{16.122}$$

and

$$\Big|\frac{\partial \delta}{\partial c_{8*}^{u;i-1}}(c_{8*}^{u;i-1}(x), T_1) - \frac{\partial \gamma}{\partial c_{8*}^{u;i-2}}(c_{8*}^{u;i-2}(x), T_1)\Big| \leq G_{c\delta}\Delta t \tag{16.123}$$

Proof By (16.8) and (16.9), we obtain straightforwardly the first pair of inequalities. The second pair of estimates follow from inequality (16.25) in Lemma 16.7, the form of the function δ and the estimates (16.50). To prove estimate (16.122) we use Lemma 16.15 and write the expression inside the mid signs $|\cdot|$ at the left hand side of (16.122) in the form

$$\frac{\partial \gamma}{\partial c^u_{p*}}(c^{u;i-1}_{1*}(x), c^{u;i-1}_{8*}(x), T_1) - \frac{\partial \gamma}{\partial c^u_{p*}}(c^{u;i-2}_{1*}(x), c^{u;i-2}_{8*}(x), T_1) =$$
$$\frac{\partial \gamma}{\partial c^u_{p*}}\Big(c^{u;i-2}_{1*}(x) + [c^{u;i-1}_{1*}(x) - c^{u;i-2}_{1*}(x)], c^{u;i-2}_{8*}(x) + [c^{u;i-1}_{8*}(x) - c^{u;i-2}_{8*}(x)], T_1\Big) -$$
$$\frac{\partial \gamma}{\partial c^u_{p*}}(c^{u;i-2}_{1*}(x), c^{u;i-2}_{8*}(x), T_1) =$$
$$\textstyle\sum_{r=1,8}[c^{u;i-1}_{r*}(x) - c^{u;i-2}_{r*}(x)] \cdot$$
$$\int_0^1 \frac{\partial^2 \gamma}{\partial c^u_{p*} \partial c^u_{r*}}\Big(c^{u;i-2}_{1*}(x) + s[c^{u;i-1}_{1*}(x) - c^{u;i-2}_{1*}(x)], c^{u;i-2}_{8*}(x) + s[c^{u;i-1}_{8*}(x) - c^{u;i-2}_{8*}(x)], T_1\Big) ds.$$

Now, using inequalities (16.50) in Lemma 16.12 and the second inequality (16.27) in Lemma 16.7, we obtain (16.122). In the same way we can obtain (16.123). □

Consequently, by (16.116), we conclude that

$$\Big|B^{i-1}_{T_1}(x; \tau^{i-1}_1(x, T_1), \tau^{i-1}_8(x, T_8)) - B^{i-2}_{T_1}(x; \tau^{i-2}_1(x, T_1), \tau^{i-2}_8(x, T_8))\Big| \leq$$
$$\Big| \big[B^{i-1}_{\tau_1}(x; \tau^{i-1}_1(x, T_1), \tau^{i-1}_8(x, T_8)) - B^{i-2}_{\tau_1}(x; \tau^{i-1}_1(x, T_1), \tau^{i-1}_8(x, T_8))\big] \cdot (1 - \gamma^{i-1}_{,T_1}\Delta t)\Big| +$$
$$\Big(\mathcal{B}_{11}G_{21} + \mathcal{B}_{11}G_{28}\Big)(\Delta t)^2 \leq |H^{i-1}_1|(1 + A_M\Delta t) + \Big(\mathcal{B}_{11}G_{21} + \mathcal{B}_{18}G_{28}\Big)(\Delta t)^2,$$

where

$$A_M := \max\{A_-, A_+\} \tag{16.124}$$

with A_-, A_+ defined in inequality (16.23) of Lemma 16.7, and A defined by (16.30).

It thus follows by means of the maximum principle that

$$|H^i_1| \leq (|H^{i-1}_1|(1 + A_M\Delta t) + W_{H_1}(\Delta t)^2 + \Delta t\,|Z^i|)L, \tag{16.125}$$

where $W_{H_1} = \mathcal{B}_{11}G_{21} + \mathcal{B}_{18}G_{28}$. In the same way, for some constant W_{H_8},

$$|H^i_8| \leq (|H^{i-1}_8|(1 + \delta_2\Delta t) + W_{H_8}\Delta t + \Delta t\,|Z^i|)L. \tag{16.126}$$

By defining, as before,

$$|H|^i := \sup\{|H^i_1|, |H^i_8|\},$$

we obtain

$$|H|^i \leq (|H^{i-1}|(1 + A_{18}\Delta t) + W_H(\Delta t)^2 + \Delta t\,|Z^i|)L, \tag{16.127}$$

where $W_H = \sup\{W_{H_1}, W_{H_8}\}$ and $A_{18} = \sup\{A_M, \delta_2\}$.

By combining (16.110) and (16.172) we obtain the system:

$$\begin{aligned} &|Z^i| \leq (|Z^{i-1}| + r_2(\Delta t)^2 + r_3|H|^{i-1}\Delta t)L. \\ &|H|^i \leq (|H^{i-1}|(1 + A_{18}\Delta t) + W_H(\Delta t)^2 + \Delta t\,|Z^i|)L. \end{aligned} \tag{16.128}$$

We will find an upper bound for $|Z|^i$ and $|H|^i$ provided by solutions to the following system of equations:

$$|Z^i| = (|Z^{i-1}| + a_*(\Delta t)^2 + a|H|^{i-1}\Delta t)L.$$
$$|H|^i = (|H^{i-1}|(1 + b\Delta t) + a_*(\Delta t)^2 + \Delta t\,|Z^i|)L, \tag{16.129}$$

where

$$a_* := \max\{W_H, r_2\}, \quad b := A_{18}.$$

In fact, we will consider the system for the differences:

$$\psi(i) := |Z^i| - |Z^{i-1}|, \quad \phi(i) := |H|^i - |H|^{i-1},$$

$i \in \{2, \dots, n\}$, which reads

$$\psi(i) = (\psi(i-1) + a\phi(i-1)d)L$$
$$\phi(i) = (\phi(i-1)(1 + bd) + \psi(i)d)L,$$

where

$$d := \Delta t.$$

Using the first equation in the second one, we can write the system in the standard form of recursive sequences:

$$\psi(i) = (\psi(i-1) + a\phi(i-1)d)L$$
$$\phi(i) = (\psi(i-1)dL + \phi(i-1)(1 + bd + ad^2L))L. \tag{16.130}$$

Note that, due to (16.113), (16.118), (16.119) and (16.129), we have:

$$\psi(2) = O(\Delta t^2), \quad \phi(2) = O(\Delta t^2).$$

If $X(i) := (\psi(i), \phi(i))^T$, then system (16.130) can be written as

$$X(i) = \mathcal{A}X(i-1), \tag{16.131}$$

with

$$\mathcal{A} = \begin{pmatrix} L & adL \\ dL^2 & (1 + bd + ad^2L)L \end{pmatrix}. \tag{16.132}$$

For $d > 0$ sufficiently small, the eigenvalues λ_1 and λ_2 of the matrix $\mathcal{A}$ are both positive and are equal to

$$\lambda_1 = \frac{1}{2}\left(cL - L\sqrt{c^2 + 4ad^2L} + 2L\right), \quad \lambda_2 = \frac{1}{2}\left(cL + L\sqrt{c^2 + 4ad^2L} + 2L\right), \tag{16.133}$$

where

$$c = bd + ad^2L.$$

The solutions to (16.131) are given by the powers of the matrix $\mathcal{A}$. To find $\mathcal{A}^n$, we will use the following result from [11].

Lemma 16.17. *(see [11, Theorem 1]) Let U be a $k \times k$ nonsingular matrix with eigenvalues $\lambda_1, \dots, \lambda_k$ and let $M(0) = I$, $M(j) = \Pi_{i=1}^{j}(U - \lambda_i I)$, $j \geq 1$. Suppose that $u_j(m)$ satisfy the (recursive) system*

$$u_1(m+1) = \lambda_1 u_1(m), \quad u_1(0) = 1$$
$$u_{j+1}(m+1) = \lambda_{j+1}u_{j+1}(m) + u_j(m), \quad u_{j+1}(0) = 0, \; j = 1, \dots, k-1.$$

Then, for $m \geq k$

$$U^m = \sum_{j=0}^{k-1} u_{j+1}(m) M(j). \tag{16.134}$$

In our case

$$M_1 = (\mathcal{A} - \lambda_1 I) = \begin{pmatrix} \frac{1}{2}\left(\sqrt{c^2 + 4ad^2 L} - c\right) L & adL \\ \\ dL^2 & \frac{1}{2}\left(c + \sqrt{c^2 + 4ad^2 L}\right) L. \end{pmatrix} \tag{16.135}$$

Now, let us note that we have

$$u_1(m) = \lambda_1^m.$$

Next, as $u_2(0) = 0$, we have

$$\begin{aligned} u_2(1) &= \lambda_2 u_2(0) + u_1(0) = 1, \\ u_2(2) &= \lambda_2 u_2(1) + u_1(1) = \lambda_2 + \lambda_1 \\ u_2(3) &= \lambda_2(\lambda_2 + \lambda_1) + \lambda_1^2 = \lambda_2^2 + \lambda_2 \lambda_1 + \lambda_1^2 \end{aligned}$$

and, in general, for $m \geq 2$,

$$u_2(m) = \sum_{s=0}^{m-1} \lambda_1^s \lambda_2^{m-1-s} = \frac{\lambda_2^m - \lambda_1^m}{\lambda_2 - \lambda_1}$$

As, $\lambda_2 > \lambda_1$, then it follows from (16.134) that

$$\mathcal{A}^m = \lambda_1^m\, I + \left(\sum_{s=0}^{m-1} \lambda_1^s \lambda_2^{m-1-s}\right) (\mathcal{A} - \lambda_1 I) < \lambda_1^m\, I + \lambda_2^{m-1}\Big(m \cdot (\mathcal{A} - \lambda_1 I)\Big) \tag{16.136}$$

where the last inequality should be understood entry-wise. By means of the identity $\sqrt{y_1 + y_2} < \sqrt{y_1} + \sqrt{y_2}$, we conclude that

$$\begin{aligned} &\frac{1}{2}\left(c - \sqrt{c^2 + 4ad^2 L}\right) L \geq -\sqrt{a_L} dL, \\ &\frac{1}{2}\left(c + \sqrt{c^2 + 4ad^2 L}\right) L \leq cL + \sqrt{a_L} dL = (b + \sqrt{a_L}) L\, d + ad^2 L, \end{aligned}$$

where $\sqrt{a_L} = \sqrt{a}\sqrt{L}$, hence, according to (16.133),

$$\begin{aligned} &L\left(1 - \sqrt{a_L} d\right) < \lambda_1 < L \\ &0 < \lambda_2 < \left(1 + d(b + \sqrt{a_L}) + ad^2\right) L. \end{aligned} \tag{16.137}$$

Recall that $L = (1 - A\Delta t)^{-1}$. Let $d = \Delta t$ be so small that $L < 2$. Then, by means of the definition of $\sqrt{a_L}$, we have

$$S = (b + \sqrt{a_L}) + ad < b + \sqrt{2}\sqrt{a} + ad.$$

If necessary, let us decrease d to the values so small that $(1 + Sd) < (1 - Sd)^{-1} < 2$. (This can be done without losing generality, because we are interested in the limit $d = \Delta t \to 0$.) It follows that ,

$$\lambda_2 = \left(1 + \frac{1}{2}\left(c + \sqrt{c^2 + 4ad^2}\right)\right) L < (1 + Sd)\, L < L(1 - dS)^{-1}.$$

Thus

$$\lim_{n\to\infty} \lambda_2^{n-1} \leq \lim_{n\to\infty} \lambda_2^{n-1} < \lim_{n\to\infty} L^n (1 - dS)^{-n}.$$

By means of Remark after (16.32), we can estimate the last limit as $\exp(AT)\exp(ST)$. Moreover, we can also find n so large that $\lambda_2^n < 9/4\exp(AT)\exp(ST)$ for $d = T/n$. Next, according to (16.135)

$$\frac{1}{d}\cdot(\mathcal{A}-\lambda_1 I) = \begin{pmatrix} \frac{1}{2}\left(\sqrt{\tilde{c}^2+4aL}-\tilde{c}\right)L & aL \\ \\ L^2 & \frac{1}{2}\left(\tilde{c}+\sqrt{\tilde{c}^2+4aL}\right)L. \end{pmatrix} \tag{16.138}$$

where $\tilde{c} = c/d = b + adL$. The entries of the last matrix stays of the order of $O(1)$ as $d \to 0$. It is thus seen that $(\mathcal{A}-\lambda_1 I)d^{-1} = (\mathcal{A}-\lambda_1 I)nT^{-1} = O(1)$ as $n \to \infty$. Likewise, as for d arbitrarily small $\lambda_1 < L$, then

$$\lim_{n\to\infty}\lambda_1^n < \lim_{n\to\infty} L^n < \exp(AT).$$

Due to Remark after (16.32) for all $n = T/d$ sufficiently large we have $\lambda_1^n < \dfrac{3}{2}\exp(AT)$. By means of (16.136), the matrix $\mathcal{A}^n$ stays uniformly bounded as $n \to \infty$ and $d = T/n \to 0$. Moreover, as the matrix $\mathcal{A}$ (given by (16.132)) satisfies the inequality $\mathcal{A} > I$ (in the sense of entries), then for $m_1 < m_2$

$$\mathcal{A}^{m_2} > \mathcal{A}^{m_1}.$$

In fact, $\psi(i)$ and $\phi(i)$ are well defined for $i \geq 2$. However, for technical reasons, we can assume additionally that $\psi(1) = O(\Delta t^2) = O(n^{-2})$ and $\phi(1) = O(\Delta t^2) = O(n^{-2})$. Hence for any $2 \leq i \leq n$,

$$(\psi(i),\phi(i))^T \leq (\psi(1),\phi(1))^T + \mathcal{A}^{i-1}(\psi(1),\phi(1))^T \leq (\psi(1),\phi(1))^T + (O(\Delta t^2), O(\Delta t^2))^T. \tag{16.139}$$

Consequently,

$$(\sum_{m=1}^{n}\psi(m), \sum_{m=1}^{n}\phi(m)) \leq n(O(\Delta t)^2), O(\Delta t)^2)) \leq (O(\Delta t), O(\Delta t)).$$

Equivalently we can consider system (16.129), which after replacing $|Z^i|$ by z_i, $|H|^i$ by h_i and inserting $|Z^i|$ into the equations for $|H|^i$ we obtain for $i \in \{2,\ldots,n\}$

$$\begin{aligned} z_i &= (z_{i-1} + a_*(\Delta t)^2 + ah_{i-1}\Delta t)L. \\ h_i &= (h_{i-1}(1 + b\Delta t + a(\Delta t)^2 L) + a_*(\Delta t)^2 + \Delta t\, z_{i-1}L + a_*(\Delta t)^3 L)L. \end{aligned} \tag{16.140}$$

This recursive system of equations can be written in the following matrix form

$$Y(i) = \mathcal{A}Y(i-1) + G, \tag{16.141}$$

where

$$Y(i) = \begin{pmatrix} z_i \\ \\ h_i \end{pmatrix}$$

$$G = \begin{pmatrix} a_* d^2 \\ \\ a_* d^2 + a_* d^3 L \end{pmatrix} \tag{16.142}$$

and $\mathcal{A}$ is given as above by (16.132). Now, by means of the Corollary 3.18 in [12], we have for $m \geq 2$:

$$Y(m) = \mathcal{A}^m Y(1) + (\sum_{r=1}^{m-1}\mathcal{A}^{m-r-1})G.$$

By the previous estimates

$$\mathcal{A}^m < \mathcal{A}^n, \quad \text{for } n > m > 1$$

in the sense of inequalities between the entries. We can thus estimate

$$\sum_{r=1}^{n-1} \mathcal{A}^{n-r-1} < n\mathcal{A}^n,$$

hence

$$\begin{aligned} &Y(n) = \mathcal{A}^n Y(1) + (\textstyle\sum_{r=1}^{n-1} \mathcal{A}^{m-r-1})G < \mathcal{A}^n Y(1) + n\mathcal{A}^n G = \\ &O(1)\cdot Y(1) + O(1)\cdot (n\, O(1/n^2)) = O(1)\cdot O(1/n) + O(1)\cdot O(1/n) = O(d). \end{aligned} \tag{16.143}$$

Thus we are in a position to formulate the main result of this section:

Lemma 16.18. *For all $i \in \{1,\dots,n\}$, the following estimates hold:*

$$\|Z^i\|_{C^0(\overline{\Omega}\times\overline{\mathbb{R}^2_+})} < G_{0Z}\,\Delta t,\ \|H_1^i\|_{C^0(\overline{\Omega}\times\overline{\mathbb{R}^2_+})} < G_{0H1}\,\Delta t \text{ and } \|H_8^i\|_{C^0(\overline{\Omega}\times\overline{\mathbb{R}^2_+})} < G_{0H8}\,\Delta t, \tag{16.144}$$

where the constants G_{0Z}, G_{0H1}, G_{0H8} are independent of i.

16.16 Estimates of $C_x^{1+\beta}$ norms of the functions R^i

By means of the results of the previous section, we will now give estimates of $C_x^{1+\beta}$ norms of R^i. This will be done by rewriting Eq. (16.5) in the form not containing the terms proportional to $(\Delta t)^{-1}$, namely as

$$\begin{aligned} &d_R\nabla^2 R^i + F_0(x)\cdot\nabla R^i - \\ &\Big\{R^i\left[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1((i-1)\Delta t, x, T_1, T_8)\right] + W^i\Big\} \ = 0, \end{aligned} \tag{16.145}$$

where, by (16.108),

$$\begin{aligned} &W^i := \frac{1}{\Delta t}\left(R^i(x,T_1,T_8) - R^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})\right) = \\ &\frac{1}{\Delta t}\left([R^i(x,T_1,T_8) - R^{i-1}(x,T_1,T_8)] + [R^{i-1}(x,T_1,T_8) - R^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})]\right) = \\ &\frac{1}{\Delta t}\left(Z^i(x,T_1,T_8) + [R^{i-1}(x,T_1,T_8) - R^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})]\right) = \\ &\frac{Z^i}{\Delta t} + \frac{1}{\Delta t}\Big(- B_1^{i-1}(x,T_1,T_8)\cdot\gamma^{i-1}(x,T_1)\Delta t - B_8^{i-1}(x,T_1,T_8)\cdot\delta^{i-1}(x,T_8)\Delta t - \\ &\textstyle\sum_{k,l=1,8}(\tau_k^{i-1} - T_k)\cdot(\tau_l^{i-1} - T_l)\displaystyle\int_0^1 (1-s)B_{\tau_k\tau_l}^{i-1}\Big(x, T_1 + s(\tau_1^{i-1} - T_1), T_8 + s(\tau_8^{i-1} - T_8)\Big)\,ds\Big). \end{aligned}$$

By Lemma 16.18, $Z^i/\Delta t < G_{0Z}$, hence by the results of sections 16.5 and 16.7, we conclude that the expression in the curly brackets is of the order of $O(1)$. It follows from Lemma 15.5, by taking $l = 2$ and the integration power p sufficiently large that

$$\|R^i\|_{W_p^2} \le C_{2p}$$

where the constants C_{2p} are uniformly bounded for all p. By using the Sobolev imbedding theorem, we conclude that for all $\beta \in (0,1)$ there exists a constant $C_{1\beta}$ independent of $i \in \{1,\dots,n\}$ such that

$$\|R^i\|_{C^{1+\beta}(\Omega)} \le C_{R1\beta}. \tag{16.146}$$

16.17 Estimates of the higher order norms of the functions $c_k^{u;i}$

Using the estimate (16.146), we will find the bound for higher order derivatives of the functions $c_k^{u;i}$.

Lemma 16.19. *Let $n \geq 3$ be fixed. Suppose that for each $i \in \{0, 1, \ldots, n\}$ and $\beta \in (0,1)$, the $C^{1+\beta}(\Omega)$ norms of the functions R^i are bounded from above uniformly with respect to i. Then, c_1^u and c_8^u are of class $C_{t,x}^{(3+\beta)/2, 3+\beta}(((i-1)\Delta t, i\Delta t) \times \Omega)$. To be more precise, there exist constants $C_1(\beta, \Omega)$, $C_8(\beta, \Omega)$, K_1 and K_8, depending on T, such that*

$$\|c_1^u\|_{C_{t,x}^{(3+\beta)/2,2+\beta}(((i-1)\Delta t, i\Delta t)\times\Omega)} \leq C_{1\Delta}(\beta,\Omega)\left[K_1 + \|c_1^u((i-1)\Delta t, \cdot)\|_{C_x^{2+\beta}(\Omega)}\right] \tag{16.147}$$

and

$$\|c_8^u\|_{C_{t,x}^{(3+\beta)/2,3+\beta}(((i-1)\Delta t, i\Delta t)\times\Omega)} \leq C_{8\Delta}(\beta,\Omega)\left[K_8 + \|c_8^u((i-1)\Delta t, \cdot)\|_{C_x^{2+\beta}(\Omega)}\right]. \tag{16.148}$$

In particular, there exists constants P and P_1 independent of i such that as $\Delta t \to 0$

$$\|c_1^u(i\Delta t, \cdot) - c_1((i-1)\Delta t, \cdot)\|_{C^0(\Omega)} \leq P\Delta t, \quad \|c_8^u(i\Delta t, \cdot) - c_8((i-1)\Delta t, \cdot)\|_{C^0(\Omega)} \leq P\Delta t, \tag{16.149}$$

$$\|c_1^u(i\Delta t, \cdot) - c_1((i-1)\Delta t, \cdot)\|_{C^1(\Omega)} \leq P_1\Delta t, \quad \|c_8^u(i\Delta t, \cdot) - c_8((i-1)\Delta t, \cdot)\|_{C^1(\Omega)} \leq P_1\Delta t \tag{16.150}$$

and for $t \in [(i-1)\Delta t, i\Delta t]$

$$\|c_1^u(t,\cdot) - c_1((i-1)\Delta t, \cdot)\|_{C^0(\Omega)} \leq P(t-(i-1)\Delta t), \quad \|c_8^u(t,\cdot) - c_8((i-1)\Delta t, \cdot)\|_{C^0(\Omega)} \leq P(t-(i-1)\Delta t) \tag{16.151}$$

together with

$$\|c_1^u(t,\cdot) - c_1((i-1)\Delta t, \cdot)\|_{C^1(\Omega)} \leq P_1(t-(i-1)\Delta t), \quad \|c_8^u(t,\cdot) - c_8((i-1)\Delta t, \cdot)\|_{C^1(\Omega)} \leq P_1(t-(i-1)\Delta t) \tag{16.152}$$

Proof The proof of the lemma follows from Lemma 15.6. Starting from the initial data $c_1^{u,0}$, $c_8^{u,0}$ belonging to $C_x^{3+\beta}(\Omega)$ class and using the fact that $R^0 \in C^{1+\beta}$ class, we obtain a $C_{t,x}^{(3+\beta)/2,3+\beta}$ solution on the set $([0, \Delta t) \times \Omega)$. Treating $c_1^{u;1}(1 \cdot \Delta t, x)$ and $c_8^{u;1}(\Delta t, x)$ as the initial data we obtain a solution of $C_{t,x}^{(3+\beta)/2,3+\beta}$ class on the set $((1 \cdot \Delta t, 2 \cdot \Delta t] \times \Omega)$. Proceeding consecutively in this way, we obtain a $C_{t,x}^{(3+\beta)/2,3+\beta}$ solution on the set $(((i-1) \cdot \Delta t, i \cdot \Delta t] \times \Omega)$ for all $i \in \{1, \ldots, n\}$, hence using the Schauder estimates, we obtain inequalities (16.147) and (16.148). As the constants K_1 and K_8 can be chosen as independent of n and i, then in view of Leray-Schauder estimates, in particular due to the fact that the time derivative of the solutions is Holder continuous, there exists a constant P such that for $\Delta t > 0$ sufficiently small, inequality (16.149) holds. Next, noting that that, according to the definition of norms in this space, the subnorm

$$\left\|\frac{\partial}{\partial t}\left(\frac{\partial^{|\alpha|}}{(\partial x)^\alpha}\right)\right\|_{C^0(\Omega)}$$

with $|\alpha| \leq 1$ is finite (see [23], Theorem IV.5.3 and Section I.1), we arrive at inequalities (16.150). $\square$

16.18 Estimates of first order derivatives of Z^i with respect to x_k

To proceed, we will analyse the equation for spatial derivatives $Z^i_{,x_k}$, $k \in \{1, 2, 3\}$. By fixing k, we will for simplicity denote

$$' := \frac{\partial}{\partial x_k}, \quad Z^i_{,x_k} := Z'^i.$$

Remark In the proof below, for each $i \in \{2, \ldots, n\}$, we will be interested in the quantity

$$\max_{k\in\{1,\ldots,dim(\Omega)\},(T_1,T_8)\in\overline{\mathbb{R}^2_+}} \Big(\sup_{x\in\Omega} \frac{\partial Z^i}{\partial x_k} \Big) = Z'^i.$$

However, for the sake of concise notation, we will the notation Z'^i, independently of k. □

By differentiating Eq.(16.102) and using Assumption 16.14 we obtain the equation:

$$\begin{aligned}
& d_R \nabla^2 Z'^i + (F_0(x)\cdot\nabla)Z'^i + (F_0'(x)\cdot\nabla)Z^i - \\
& \frac{Z'^i(x;T_1,T_8)}{\Delta t} + \frac{Z'^{i-1}(x,T_1,T_8)}{\Delta t} + \frac{Z_*'^{i-1}(x,T_1,T_8)}{\Delta t} - \\
& \Big\{ Z'^i \Big[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1((i-1)\Delta t, x, T_1, T_8)\Big] + \\
& Z^i \Big[\Big(\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1((i-1)\Delta t, x, T_1, T_8)\Big]' \Big\} - \\
& (R^{i-1}\Delta V^i)' \ = 0
\end{aligned} \tag{16.153}$$

where

$$Z_*'^{i-1} := (R_*^{i-1} - R^{i-1})' - (R_*^{i-2} - R^{i-2})' \tag{16.154}$$

with R_*^{i-1} defined in (16.104) and ΔV is given by (16.107).

To begin with, let us note that according to (16.107) and (16.150) the term $(R^{i-1}\Delta V^i)'$ is of the order of $O(\Delta t)$. We are going to construct a recurrent sequence for $Z'^i - Z'^{i-1}$ and prove that these differences are of the order of $O(\Delta t)^2$. First, adapting the analysis of the term Z_*^i in Eq.(16.102), let us consider the term $Z_*'^{i-1}$. For $i \geq 2$, we have:

$$\begin{aligned}
& R'^{i-1} - R_*'^{i-1} = R'^{i-1}(x,T_1,T_8) - R'^{i-1}(x, \tau_1^{i-1}(x,T_1), \tau_8^{i-1}(x,T_8)) = \\
& R'^{i-1}(x,T_1,T_8) - R'^{i-1}(x,T_1,T_8) - \\
& B_1'^{i-1}(x,T_1,T_8)\cdot\gamma^{i-1}(x,T_1)\Delta t - B_8'^{i-1}(x,T_1,T_8)\cdot\delta^{i-1}(x,T_8)\Delta t - \\
& B_1^{i-1}(x,T_1,T_8)\cdot(\gamma^{i-1}(x,T_1))'\Delta t - B_8^{i-1}(x,T_1,T_8)\cdot(\delta^{i-1}(x,T_8))'\Delta t - \\
& \Big[(\tau_1^{i-1}-T_1)^2 \textstyle\int_0^1 (1-s)(B_{\tau_1\tau_1}^{i-1}(x, T_1 + s(\tau_1^{i-1}-T_1), T_8 + s(\tau_8^{i-1}-T_8))\,ds + \\
& (\tau_8^{i-1}-T_8)^2 \textstyle\int_0^1 (1-s)(B_{\tau_8\tau_8}^{i-1}(x, T_1 + s(\tau_1^{i-1}-T_1), T_8 + s(\tau_8^{i-1}-T_8))\,ds + \\
& 2(\tau_1^{i-1}-T_1)(\tau_8^{i-1}-T_8) \textstyle\int_0^1 (1-s)(B_{\tau_1\tau_8}^{i-1}(x, T_1 + s(\tau_1^{i-1}-T_1), T_8 + s(\tau_8^{i-1}-T_8))\,ds\Big]'.
\end{aligned} \tag{16.155}$$

Next, by means of section 16.13 and Lemma 16.16

$$\begin{aligned}
& B_1'^{i-1}(x,T_1,T_8)\cdot\gamma^{i-1}(x,T_1)\Delta t - B_1'^{i-2}(x,T_1,T_8)\cdot\gamma^{i-2}(x,T_1)\Delta t = \\
& \Big(B_1'^{i-1}(x,T_1,T_8) - B_1'^{i-2}(x,T_1,T_8)\Big)\cdot\gamma^{i-1}(x,T_1)\Delta t + B_1'^{i-2}(x,T_1,T_8)\Big(\gamma^{i-1}(x,T_1) - \gamma^{i-2}(x,T_1)\Big)\Delta t := \\
& H_1'^{i-1}(x,T_1,T_8)\cdot\gamma^{i-1}(x,T_1)\Delta t + B_1'^{i-2}(x,T_1,T_8)\Big(\gamma^{i-1}(x,T_1) - \gamma^{i-2}(x,T_1)\Big)\Delta t = \\
& H_1'^{i-1}(x,T_1,T_8)\cdot\gamma^{i-1}(x,T_1)\Delta t + \mathcal{Q}_{*,*}^{i-2} O(\Delta t)\Delta t
\end{aligned} \tag{16.156}$$

and

$$
\begin{aligned}
& B_1^{i-1}(x,T_1,T_8)\cdot(\gamma^{i-1}(x,T_1))'\Delta t - B_1^{i-2}(x,T_1,T_8)\cdot(\gamma^{i-2}(x,T_1))'\Delta t = \\
& \Big(B_1^{i-1}(x,T_1,T_8) - B_1^{i-2}(x,T_1,T_8)\Big)\cdot(\gamma^{i-1}(x,T_1))'\Delta t + B_1^{i-2}(x,T_1,T_8) \\
& \cdot\Big((\gamma^{i-1}(x,T_1))' - (\gamma^{i-2}(x,T_1))'\Big)\Delta t \\
& = H_1^{i-1}(x,T_1,T_8)\cdot\gamma^{i-1}(x,T_1)\Delta t + B_1^{i-2}(x,T_1,T_8)\cdot\Big((\gamma^{i-1}(x,T_1))' - (\gamma^{i-2}(x,T_1))'\Big)\Delta t
\end{aligned}
\tag{16.157}
$$

where, for $' = \dfrac{\partial}{\partial x_k}$,

$$
\begin{aligned}
& (\gamma^{i-1}(x,T_1))' - (\gamma^{i-2}(x,T_1))' = \textstyle\sum_{p=1,8}\Big\{\dfrac{\partial\gamma}{\partial c_{p*}^{u;i-1}}\cdot\dfrac{\partial c_{p*}^{u;i-1}}{\partial x_k} - \dfrac{\partial\gamma}{\partial c_{p*}^{u;i-2}}\cdot\dfrac{\partial c_{p*}^{u;i-2}}{\partial x_k}\Big\} = \\
& \textstyle\sum_{p=1,8}\Big\{\Big[\dfrac{\partial\gamma}{\partial c_{p*}^{u;i-1}} - \dfrac{\partial\gamma}{\partial c_{p*}^{u;i-2}}\Big]\cdot\dfrac{\partial c_{p*}^{u;i-1}}{\partial x_k} + \dfrac{\partial\gamma}{\partial c_{p*}^{u;i-2}}\cdot\Big[\dfrac{\partial c_{p*}^{u;i-1}}{\partial x_k} - \dfrac{\partial c_{p*}^{u;i-2}}{\partial x_k}\Big]\Big\}.
\end{aligned}
\tag{16.158}
$$

Likewise, by means of section 16.13 and Lemma 16.16

$$
\begin{aligned}
& B_8'^{i-1}(x,T_1,T_8)\cdot\delta^{i-1}(x,T_1)\Delta t - B_8'^{i-2}(x,T_1,T_8)\cdot\delta^{i-2}(x,T_1)\Delta t = \\
& \Big(B_8'^{i-1}(x,T_1,T_8) - B_8'^{i-2}(x,T_1,T_8)\Big)\cdot\delta^{i-1}(x,T_1)\Delta t + B_1'^{i-2}(x,T_1,T_8)\Big(\delta^{i-1}(x,T_1) - \delta^{i-2}(x,T_1)\Big)\Delta t \\
& := H_8'^{i-1}(x,T_1,T_8)\cdot\delta^{i-1}(x,T_1)\Delta t + B_8'^{i-2}(x,T_1,T_8)\Big(\delta^{i-1}(x,T_1) - \delta^{i-2}(x,T_1)\Big)\Delta t \\
& = H_8'^{i-1}(x,T_1,T_8)\cdot\delta^{i-1}(x,T_1)\Delta t + \mathcal{Q}_{*,*}^{i-2}O(\Delta t)\Delta t
\end{aligned}
\tag{16.159}
$$

and

$$
\begin{aligned}
& B_8^{i-1}(x,T_1,T_8)\cdot(\delta^{i-1}(x,T_8))'\Delta t - B_8^{i-2}(x,T_1,T_8)\cdot(\delta^{i-2}(x,T_8))'\Delta t = \\
& \Big(B_8^{i-1}(x,T_1,T_8) - B_8^{i-2}(x,T_1,T_8)\Big)\cdot(\delta^{i-1}(x,T_8))'\Delta t + B_8^{i-2}(x,T_1,T_8) \\
& \qquad\cdot\Big((\delta^{i-1}(x,T_8))' - (\delta^{i-2}(x,T_8))'\Big)\Delta t \\
& = H_8^{i-1}(x,T_1,T_8)\cdot\delta^{i-1}(x,T_8)\Delta t + B_8^{i-2}(x,T_1,T_8)\cdot\Big((\delta^{i-1}(x,T_1))' - (\delta^{i-2}(x,T_1))'\Big)\Delta t
\end{aligned}
\tag{16.160}
$$

The first term of the right hand side of (16.157), due to Lemma 16.18, can be estimated by a constant (independent of i) times $(\Delta t)^2$. By the results of section 16.8, $|B_1^{i-2}(x,T_1,T_8)|$ is bounded uniformly with i. Next, by (16.122) in Lemma 16.16, the first square bracket at the right hand side of (16.158) is of the order of $O(\Delta t)$. Similarly, due to estimate (16.150) in Lemma 16.19, the second square bracket in (16.158) is of the order of $O(\Delta t)$. Similar conclusions can be drawn with respect to the expression given by the right hand side of (16.160). Finally, as $(\tau_1^{i-1} - T_1) = \gamma\Delta t$ and $(\tau_8^{i-1} - T_8) = \delta\Delta t$, in view of Lemma 16.12 (differentiability of the functions γ and δ) and section 16.14 (differentiability with respect to x_k of $B_{\tau_k\tau_l}$), we conclude that there exists a constant r_1 independent of i, such that for all $x \in \overline{\Omega}$ and $(T_1,T_8) \in \overline{\mathbb{R}_+^2}$,

$$
|(R_*'^{i-1} - R'^{i-1}) - (R_*'^{i-2} - R'^{i-2})| = \Big(|H_1'^{i-1}|\overline{\gamma} + |H_8'^{i-1}|\overline{\delta}\Big)\Delta t + r_1(\Delta t)^2.
$$

Suppose that

$$\|\nabla Z^i\| = \Big|\frac{\partial Z^i}{\partial x_k}(x_*)\Big| := Z'^i(x_*).$$

Then, by applying to Eq. (16.153) the maximum principle at the point x_*, we can estimate for Δt sufficiently small

$$|Z'^i(x_*)| \le \Big\{|Z'^{i-1}(x_*)|\frac{1}{\Delta t} + \Big(|H_1'^{i-1}|\overline{\gamma}\Delta t + |H_8'^{i-1}|\overline{\delta}\Delta t + r_1(\Delta t)^2\Big)\frac{1}{\Delta t} + r_4\Delta t\Big\}\Big(\frac{1}{\Delta t} - A - 3f_0\Big)^{-1}.$$

Denoting, similarly to (16.161),

$$|H'|^i := \sup\{|H_1'^i|, |H_8'^i|\}, \tag{16.161}$$

we arrive at the inequality corresponding to (16.110):

$$|Z'^i| \le (|Z'^{i-1}| + \tilde{r}_2(\Delta t)^2 + \tilde{r}_3|H'|^{i-1}\Delta t)L, \tag{16.162}$$

where

$$L = (1 - A_1\Delta t), \quad A_1 = A + 3f_0. \tag{16.163}$$

(Cf. (16.82).) Let us note that by differentiating Eq. (16.112) with respect to x_k, we can prove that

$$\|\nabla Z_1(\cdot)\|_{C^0(\Omega)} < \Delta t\, G_Z^1, \tag{16.164}$$

for some constant G_Z^1.

In the similar way, we can derive the equation for the components of ∇H_1^i for $i \in \{2, \dots, n\}$. Let, similarly as before:

$$' := \frac{\partial}{\partial x_k}, \quad H^i_{1,x_k} := H_1'^i, \quad H^i_{8,x_k} := H_8'^i,$$

where the index k is in general a function of i. The equation for H'^i has the form:

$$\begin{aligned}
&0 = d_R\nabla^2 H_1'^i + (F_0(x)\cdot\nabla)H_1'^i + (F_0'(x)\cdot\nabla)H_1^i - \\
&H_1'^i\Big[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1((i-1)\Delta t, x, T_1, T_8) + \frac{1}{\Delta t}\Big] - \\
&B_1'^{i-1}[\Delta V_i] - B_1^{i-1}[\Delta V_i]' - Z'^i\Big[\frac{\partial^2}{\partial T_1^2}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_1}F_1((i-1)\Delta t, x, T_1, T_8)\Big] - \\
&Z^i\Big[\frac{\partial^2}{\partial T_1^2}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_1}F_1((i-1)\Delta t, x, T_1, T_8)\Big]' - \\
&\Big(R^{i-1}\Big[\frac{\partial^2}{\partial T_1^2}\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) - (\frac{\partial^2}{\partial T_1^2}\gamma(c_{1*}^{u;i-2}, c_{8*}^{u;i-2}, T_1)) + \\
&\frac{\partial}{\partial T_1}F_1((i-1)\Delta t, x, T_1, T_8) - \frac{\partial}{\partial T_1}F_1((i-2)\Delta t, x, T_1, T_8)\Big]\Big)' + \\
&\frac{\big(B_1^{i-1}(x;\tau_1^{i-1}(x,T_1), \tau_8^{i-1}(x,T_8)) - B_1^{i-2}(x;\tau_1^{i-2}(x,T_1), \tau_8^{i-2}(x,T_8))\big)'}{\Delta t}.
\end{aligned} \tag{16.165}$$

This equation can be obtained formally by differentiating (with respect to x_k) Eq.(16.114).

In accordance with (16.105) and (16.106), let us denote:

$$\gamma_{T_1}^i := \frac{\partial\gamma(c_{1*}^{u;i}, c_{8*}^{u,i}, T_1)}{\partial T_1}, \quad \delta_{T_8}^i := \frac{\partial\delta(c_{8*}^{u,i}, T_1)}{\partial T_8}.$$

We have:

$$\Big(B_1^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_1^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))\Big)' =$$

$$\Big(B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))(1-\gamma_{T_1}^{i-1}\Delta t) - B_{\tau_1}^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))(1-\gamma_{T_1}^{i-2}\Delta t)\Big)' =$$

$$\Big(\ \big[B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_{\tau_1}^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))\big]\cdot(1-\gamma_{T_1}^{i-1}\Delta t) -$$

$$B_{\tau_1}^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))\cdot\big[\gamma_{T_1}^{i-1}\Delta t - \gamma_{T_1}^{i-2}\Delta t\big]\ \Big)' \tag{16.166}$$

Let us note that, according to Lemma 16.19,

$$\Big(\gamma_{T_1}^{i-1}(x,T_1)\Delta t - \gamma_{T_1}^{i-2}(x,T_1)\Delta t\Big)' = (\gamma_{T_1}'^{i-1}(x,T_1) - \gamma_{T_1}'^{i-2})\Delta t < G_{11g}(\Delta t)^2.$$

Next, by (16.16),

$$\Big(B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_{\tau_1}^{i-2}(x;\tau_1^{i-2}(x,T_1),\tau_8^{i-2}(x,T_8))\Big)' =$$

$$\Big(B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) -$$

$$B_{\tau_1}^{i-2}\Big[x;\tau_1^{i-1}(x,T_1) - \{\tau_1^{i-1}(x,T_1) - \tau_1^{i-2}(x,T_1)\},\tau_8^{i-1}(x,T_8) - \big\{\tau_8^{i-1}(x,T_8) - \tau_8^{i-2}(x,T_8)\big\}\ \Big]\Big)' =$$

$$\Big(B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_{\tau_1}^{i-2}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) -$$

$$\Big[(\tau_1^{i-1}(x,T_1) - \tau_1^{i-2}(x,T_1))\int_0^1 (B_{\tau_1\tau_1}^{i-2}(x,\tau_1^{i-1} + s(\tau_1^{i-2} - \tau_1^{i-1}),\tau_8 + s(\tau_8^{i-1} - \tau_8^{i-2}))\,ds +$$

$$(\tau_8^{i-1}(x,T_8) - \tau_8^{i-2}(x,T_8))\int_0^1 (B_{\tau_1\tau_8}^{i-2}(x,\tau_1^{i-1} + s(\tau_1^{i-2} - \tau_1^{i-1}),\tau_8^{i-1} + s(\tau_8^{i-2} - \tau_8^{i-1})\,ds\ \Big]\Big)'. \tag{16.167}$$

If $' = \dfrac{\partial}{\partial x_k}$, we have

$$\Big(B_{\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_{\tau_1}^{i-2}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))\Big)' = H_1'^{i}(x,\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))$$

$$+\Big(B_{\tau_1\tau_1}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_{\tau_1\tau_1}^{i-2}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))\Big)\cdot\frac{\partial\tau_1^{i-1}(x,T_1)}{\partial x_k}$$

$$+\Big(B_{\tau_1\tau_8}^{i-1}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8)) - B_{\tau_1\tau_8}^{i-2}(x;\tau_1^{i-1}(x,T_1),\tau_8^{i-1}(x,T_8))\Big)\cdot\frac{\partial\tau_8^{i-1}(x,T_1)}{\partial x_k} \tag{16.168}$$

Remark Recall that $\dfrac{\partial\tau_1^{i-1}(x,T_1)}{\partial x_k}$ and $\dfrac{\partial\tau_8^{i-1}(x,T_1)}{\partial x_k}$ are of the order of (Δt) as $\Delta t \to 0$ (see Lemma 16.3). It follows that the second and the third term in the above expression is of the order of $O((\Delta t)^2)$, if only the coefficients multiplying $\dfrac{\partial\tau_1^{i-1}(x,T_1)}{\partial x_k}$ and $\dfrac{\partial\tau_8^{i-1}(x,T_1)}{\partial x_k}$ are of the order of Δt. □

Next,

$$\Big[(\tau_1^{i-1}(x,T_1)-\tau_1^{i-2}(x,T_1))\cdot$$

$$\int_0^1 (B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-1}(x,T_1)+s(\tau_1^{i-2}(x,T_1)-\tau_1(x,T_1)),\tau_8(x,T_8)+s(\tau_8^{i-1}(x,T_8)-\tau_8^{i-2}(x,T_8)))\,ds\Big]'=$$

$$(\tau_1^{i-1}(x,T_1)-\tau_1^{i-2}(x,T_1))'\cdot$$

$$\int_0^1 (B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-1}(x,T_1)+s(\tau_1^{i-2}(x,T_1)-\tau_1(x,T_1)),\tau_8(x,T_8)+s(\tau_8^{i-1}(x,T_8)-\tau_8^{i-2}(x,T_8)))\,ds+$$

$$(\tau_1^{i-1}(x,T_1)-\tau_1^{i-2}(x,T_1))\cdot$$

$$\int_0^1 \Big(B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-1}(x,T_1)+s(\tau_1^{i-2}(x,T_1)-\tau_1(x,T_1)),\tau_8(x,T_8)+s(\tau_8^{i-1}(x,T_8)-\tau_8^{i-2}(x,T_8)))\Big)'\,ds$$

Let us note that, in view of Lemma 16.16 and Lemma 16.19,

$$(\tau_1^{i-1}(x,T_1)-\tau_1^{i-2}(x,T_1)=O((\Delta t)^2),\quad \text{and}\quad (\tau_1^{i-1}(x,T_1)-\tau_1^{i-2}(x,T_1))'=O((\Delta t)^2)$$

as $\Delta t\to 0$. On the other hand the quantity

$$\Big(B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-1}(x,T_1)+s(\tau_1^{i-2}(x,T_1)-\tau_1(x,T_1)),\tau_8(x,T_8)+s(\tau_8^{i-1}(x,T_8)-\tau_8^{i-2}(x,T_8)))\Big)'$$

is finite, if only the third order derivatives $B^k_{\tau_1\tau_1 x}(x,\tau_1,\tau_8)$, $B^k_{\tau_1\tau_1\tau_1}(x,\tau_1,\tau_8)$, $B^k_{\tau_1\tau_1\tau_8}(x,\tau_1,\tau_8)$ are bounded for all the possible x, τ_1 and τ_8 of interest. Likewise,

$$\Big(B^{i-2}_{18}(x,\tau_1^{i-1}+s(\tau_1^{i-2}-\tau_1^{i-1}),\tau_8^{i-1}+s(\tau_8^{i-2}-\tau_8^{i-1})\Big)'$$

is finite, if only the third order derivatives $B^k_{\tau_1\tau_1 x}(x,\tau_1,\tau_8)$, $B^k_{\tau_1\tau_8\tau_1}(x,\tau_1,\tau_8)$, $B^k_{\tau_1\tau_8\tau_8}(x,\tau_1,\tau_8)$ are bounded for all the possible x, T_1 and T_8 of interest.

In view of Remark after (16.168), we have to estimate the differences

$$H^i_{km}:=B^i_{km}(x,T_1,T_8)-B^{i-1}_{km}(x,T_1,T_8),\quad k,m\in\{1,8\}.$$

Let us consider the difference H^i_{11}. The remaining differences (H^i_{18} and H^i_{88}) can be considered similarly. The equation for H^i_{11} is obtained by subtracting the equation (16.85) for B^{i-1}_{11} from the equation for B^i_{11}. We have

$$
\begin{aligned}
&0 = d_R \nabla^2 H_{11}^i + F_0((i-1)\Delta t, x) \cdot \nabla H_{11}^i - \\
&H_{11}^i \left[\frac{\partial}{\partial T_1} \Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) \Big) + \frac{\partial}{\partial T_8} \Big(\delta(c_{8*}^{u;i-1}, T_8) \Big) + F_1((i-1)\Delta t, x, T_1, T_8) + \frac{1}{\Delta t} \right] - \\
&B_{11}^{i-1} \Big\{ \left[\frac{\partial}{\partial T_1} \Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) \Big) + \frac{\partial}{\partial T_8} \Big(\delta(c_{8*}^{u;i-1}, T_8) \Big) + F_1((i-1)\Delta t, x, T_1, T_8) \right] - \\
&\qquad \left[\frac{\partial}{\partial T_1} \Big(\gamma(c_{1*}^{u;i-2}, c_{8*}^{u;i-2}, T_1) \Big) + \frac{\partial}{\partial T_8} \Big(\delta(c_{8*}^{u;i-2}, T_8) \Big) + F_1((i-2)\Delta t, x, T_1, T_8) \right] \Big\}_1 - \\
&2B_1^i \Big\{ \left[\frac{\partial^2}{\partial T_1^2} \Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) \Big) + \frac{\partial F_1}{\partial T_1}((i-1)\Delta t, x, T_1, T_8) \right] - \\
&\qquad \left[\frac{\partial^2}{\partial T_1^2} \Big(\gamma(c_{1*}^{u;i-2}, c_{8*}^{u;i-2}, T_1) \Big) + \frac{\partial F_1}{\partial T_1}((i-2)\Delta t, x, T_1, T_8) \right] \Big\}_2 - \\
&R^i \Big\{ \Big[\frac{\partial^3}{\partial T_1^3} \Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) \Big) + \frac{\partial^2 F_1}{\partial T_1^2}((i-1)\Delta t, x, T_1, T_8) \Big] - \\
&\qquad \Big[\frac{\partial^3}{\partial T_1^3} \Big(\gamma(c_{1*}^{u;i-2}, c_{8*}^{u;i-2}, T_1) \Big) + \frac{\partial^2 F_1}{\partial T_1^2}((i-2)\Delta t, x, T_1, T_8) \Big] \Big\}_3 - \\
&\quad Z^i \Big[\frac{\partial^3}{\partial T_1^3} \Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1) \Big) + \frac{\partial^2 F_1}{\partial T_1^2}((i-1)\Delta t, x, T_1, T_8) \Big] + \\
&\frac{1}{\Delta t} \Big[B_{T_1 T_1}^{i-1}(x; \tau_1^{i-1}(x, T_1), \tau_8^{i-1}(x, T_8)) - B_{T_1 T_1}^{i-2}(x; \tau_1^{i-2}(x, T_1), \tau_8^{i-2}(x, T_8)) \Big]_4 .
\end{aligned}
\tag{16.169}
$$

Recall that according to (16.86):

$$
\begin{aligned}
&B_{11}^{i-1}(x, \tau_1^{i-1}, \tau_8^{i-1}) = B_{11}^{i-1}(x, \tau_1^{i-1}(x, T_1), \tau_8^{i-1}(x, T_8)) = \\
&B_{\tau_1 \tau_1}^{i-1}(x, \tau_1^{i-1}, \tau_8^{i-1}) \cdot \left(1 - \Delta t \, \frac{\partial \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial T_1} \right)^2 - \\
&\Delta t \; B_{\tau_1}^{i-1}(x, \tau_1^{i-1}, \tau_8^{i-1}) \cdot \frac{\partial^2 \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial T_1^2} .
\end{aligned}
\tag{16.170}
$$

Let us note that the expressions in the curly brackets $\{\cdot\}_1$, $\{\cdot\}_2$ and $\{\cdot\}_3$ in (16.169) are of the order $O(1)\Delta t$. Using (16.170) in the square bracket $[\cdot]_4$, we have

$$
\begin{aligned}
&[\cdot]_4 = \Big(B_{\tau_1 \tau_1}^{i-1}(x, \tau_1^{i-1}, \tau_8^{i-1}) - B_{\tau_1 \tau_1}^{i-2}(x, \tau_1^{i-2}, \tau_8^{i-2}) \Big) \cdot \left(1 - \Delta t \, \frac{\partial \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial T_1} \right)^2 - \\
&B_{\tau_1 \tau_1}^{i-2}(x, \tau_1^{i-2}, \tau_8^{i-2}) \cdot \Big(2 - \Delta t \, \frac{\partial \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial T_1} - \Delta t \, \frac{\partial \gamma(c_{1*}^{u;i-2}(x), c_{8*}^{u;i-2}(x), T_1)}{\partial T_1} \Big) \cdot \\
&\Delta t \Big(\frac{\partial \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial T_1} - \frac{\partial \gamma(c_{1*}^{u;i-2}(x), c_{8*}^{u;i-2}(x), T_1)}{\partial T_1} \Big) - \\
&\Delta t \, \Big(B_{\tau_1}^{i-1}(x, \tau_1^{i-1}, \tau_8^{i-1}) - B_{\tau_1}^{i-2}(x, \tau_1^{i-2}, \tau_8^{i-2}) \Big) \cdot \frac{\partial^2 \gamma(c_{1*}^{u;i-1}(x), c_{8*}^{u;i-1}(x), T_1)}{\partial T_1^2} - \\
&\Delta t \, B_{\tau_1}^{i-2}(x, \tau_1^{i-2}, \tau_8^{i-2}) \cdot \Big(\frac{\partial^2 \gamma(c_{1*}^{u;i-2}(x), c_{8*}^{u;i-2}(x), T_1)}{\partial T_1^2} - \frac{\partial^2 \gamma(c_{1*}^{u;i-2}(x), c_{8*}^{u;i-2}(x), T_1)}{\partial T_1^2} \Big)
\end{aligned}
\tag{16.171}
$$

The second, the third and the fourth term in (16.171) are of the order of $(\Delta t)^2$ (as $\Delta t \to 0$). Next, taking advantage of the fact that, according to section 16.12, the third order derivatives $B_{\tau_1 \tau_1 \tau_1}^i(x, \tau_1, \tau_8)$ and $B_{\tau_1 \tau_1 \tau_8}^i(x, \tau_1, \tau_8)$ are bounded independently of $x \in \overline{\Omega}$ and all $(T_1, T_8) \geq 0$, we obtain:

$B^{i-1}_{\tau_1\tau_1}(x,\tau_1^{i-1},\tau_8^{i-1}) - B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-2},\tau_8^{i-2}) =$

$\Big(B^{i-1}_{\tau_1\tau_1}(x,\tau_1^{i-1},\tau_8^{i-1}) - B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-1},\tau_8^{i-1})\Big) + \Big(B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-1},\tau_8^{i-1}) - B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-2},\tau_8^{i-2})\Big) =$

$\Big(B^{i-1}_{\tau_1\tau_1}(x,\tau_1^{i-1},\tau_8^{i-1}) - B^{i-2}_{\tau_1\tau_1}(x,\tau_1^{i-1},\tau_8^{i-1})\Big)+$

$\Big[(\tau_1^{i-1}(x,T_1) - \tau_1^{i-2}(x,T_1))\int_0^1 (B^{i-1}_{11}(x,T_1+s(\tau_1^{i-1}(x,T_1)-T_1),T_8+s(\tau_8^{i-1}(x,T_8)-T_8))\,ds+$

$(\tau_8^{i-1}(x,T_8) - \tau_8^{i-2}(x,T_8))\int_0^1 (B^{i-1}_{88}(x,T_1+s(\tau_1^{i-1}(x,T_1)-T_1),T_8+s(\tau_8^{i-1}(x,T_8)-T_8))\,ds\Big]$

Using (16.171), we obtain by means of the maximum principle that for some constant A_{*11}

$$|H^i_{11}| \le |H^{i-1}_{11}|(1-A\Delta t)^{-1} + A_{*11}(\Delta t)^2.$$

This recursive relation can be written formally as:

$$Y(i) = \mathcal{A}Y(i-1) + G,$$

where

$$Y(i) = |H^i_{11}|, \quad G = A_{*11}(\Delta t)^2\,,$$

$\mathcal{A}$ is given by (16.132) with $L = (1-A\Delta t)^{-1}$ and A is defined in (16.30).

Let us note that by taking $i=1$ in (16.169) and assuming sufficiently smooth initial conditions R^0, c_1^u and c_8^u it is seen that $|H^1_{11}| = O(\Delta t)$. Similarly by considering the equations corresponding to (16.169) for H^1_{18} and H^1_{88}, we conclude that $|H^1_{18}| = O(\Delta t)$ and $|H^1_{88}| = O(\Delta t)$.

Now, by means of the Corollary 3.18 in [12], we have for $m \ge 2$:

$$Y(m) = \mathcal{A}^m Y(1) + (\sum_{r=1}^{m-1}\mathcal{A}^{m-r-1})G.$$

By the previous estimates

$$\mathcal{A}^s < \mathcal{A}^{m-1}, \quad \text{for } m-1 > s \ge 1$$

in the sense of inequalities between the entries. Taking into account that $n = T(\Delta t)^{-1}$, we can thus estimate

$$G\sum_{r=1}^{m-1}\mathcal{A}^{m-r-1} < G\,(m-1)\mathcal{A}^{m-1} < G\,n\,\mathcal{A}^{m-1} = A_{*11}T\Delta t\,\mathcal{A}^{m-1} < A_{*11}T\Delta t\,\mathcal{A}^m,$$

hence for $\Delta t > 0$ sufficiently small and all $i \in \{1,\dots,n\}$, using Remark after Lemma 16.8,

$$|H^i_{11}| \le Y(i) \le 3/2\exp(Ai\Delta t)H^1_{11} + 3/2\,\Delta t\,A_{*11}T\,\exp(Ai\Delta t) \underset{\Delta t\to 0}{\longrightarrow} 3/2\exp(Ai\Delta t)H^1_{11}.$$

In the same way we can show that

$$|H^i_{88}| \le 3/2\exp(Ai\Delta t)H^1_{88} \underset{\Delta t\to 0}{\longrightarrow} 3/2\exp(Ai\Delta t)H^1_{88}$$

and

$$|H^n_{18}| \le 3/2\exp(Ai\Delta t)H^1_{18} \underset{\Delta t\to 0}{\longrightarrow} 3/2\exp(Ai\Delta t)H^1_{18}.$$

Assuming that the derivatives B^j_{111} and B^j_{118} are bounded and continuous (independently of $j \in \{1,\dots,n\}$, we conclude that

$$\Big[B_{11}^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1}) - B_{11}^{i-2}(x;\tau_1^{i-2},\tau_8^{i-2})\Big] = \Big(B_{11}^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1}) - B_{11}^{i-2}(x;\tau_1^{i-1},\tau_8^{i-1})\Big) +$$

$$\Big[(\tau_1^{i-1}(x,T_1) - \tau_1^{i-2}(x,T_1)) \int_0^1 (B_{111}^{i-1}(x, T_1 + s(\tau_1^{i-1}(x,T_1) - T_1), T_8 + s(\tau_8^{i-1}(x,T_8) - T_8))\, ds +$$

$$(\tau_8^{i-1}(x,T_8) - \tau_8^{i-2}(x,T_8)) \int_0^1 (B_{118}^{i-1}(x, T_1 + s(\tau_1^{i-1}(x,T_1) - T_1), T_8 + s(\tau_8^{i-1}(x,T_8) - T_8))\, ds\Big].$$

Taking into account the above estimates, and carrying out a similar analysis for the equation for $H_8'^i$, we can deduce the inequalities:

$$|H'|^i \le (|H'|^{i-1}(1 + A_{g18}\Delta t) + W_{gH}(\Delta t)^2 + \Delta t\, |Z'^i|)L, \tag{16.172}$$

for some constants A_{g18} and W_{gH} independent of i.

Next, by differentiating Eq.(16.118) and using similar arguments, we conclude that:

$$\left|H_1'^1\right| < h_{g1}^1 \Delta t \quad \text{and} \quad \left|H_8'^1\right| < h_{g8}^1 \Delta t \tag{16.173}$$

hence

$$\max\left\{\left|H_1'^1\right|, \left|H_8'^1\right|\right\} \le \max\{h_{g1}^1, h_{g8}^1\}\Delta t := \mathcal{H}^1 \Delta t.$$

Now, proceeding, like in section 16.15, either using the scheme of the form (16.131) or the scheme (16.141), we obtain estimates corresponding to the estimates (16.144). Let us use the scheme corresponding to (16.141).

Putting (16.162) into (16.172), we obtain a pair of inequalities:

$$\begin{aligned} &|H'|^i \le (|H'|^{i-1}(1 + A_{g18}\Delta t + \tilde{r}_3(\Delta t)^2 L) + W_{gH}(\Delta t)^2 + \Delta t\, |Z'^{i-1}| L + \tilde{r}_2(\Delta t)^3 L)L, \\ &|Z'^i| \le (|Z'^{i-1}| + \tilde{r}_2(\Delta t)^2 + \tilde{r}_3 |H'|^{i-1}\Delta t)L. \end{aligned} \tag{16.174}$$

Replacing $|Z^i|$ by $\mathcal{Z}_i$, $|H|^i$ by $\mathcal{H}_i$, we obtain, for $i \in \{2, \ldots, n\}$, as in the case of system (16.140),

$$\begin{aligned} &\mathcal{Z}_i = (\mathcal{Z}_{i-1} + a_* d^2 + a\mathcal{H}_{i-1} d)L. \\ &\mathcal{H}_i = (\mathcal{H}_{i-1}(1 + bd + ad^2 L) + a_*(\Delta t)^2 + d\,\mathcal{Z}_{i-1} L + a_* d^3 L)L, \end{aligned} \tag{16.175}$$

where we denoted $d = \Delta t$, with the obvious identification of the constants a_*, a, b, which are in general different than the corresponding constants for system (16.140), but have been denoted similarly for simplicity. As above, L is given by (16.163). This recursive system of equations can be written in the following matrix form

$$\mathcal{Y}(i) = \mathcal{A}\mathcal{Y}(i-1) + G, \tag{16.176}$$

where

$$\mathcal{Y}(i) = \begin{pmatrix} \mathcal{Z}_i \\ \mathcal{H}_i \end{pmatrix}$$

and

$$\mathcal{Y}(1) \le \begin{pmatrix} G_Z^1 \\ \mathcal{H}^1 \end{pmatrix} \Delta t.$$

The matrix $\mathcal{A}$ has formally the form given by (16.132), whereas G the form given by (16.142). Repeating the analysis of system (16.140), we can show the validity of the lemma below.

Lemma 16.20. *The following estimates hold:*

$$\|\nabla Z^i\| < G_Z\,\Delta t,\ \ \|\nabla H_1^i\| < G_{H1}\,\Delta t \text{ and } \|\nabla H_8^i\| < G_{H8}\,\Delta t \tag{16.177}$$

for some constants G_Z, G_{H1} and G_{H8} independent of i.

16.19 Estimates of $C_x^{2+\beta}$ norms of the functions R^i

By means of the above results, we can now derive an 'a priori' $C^{2+\beta}$ estimate of the functions R^i. As in section 16.16, this will be done by rewriting the equation (16.5) in the form

$$\begin{aligned}
&d_R\nabla^2 R^i + F_0((i-1)\Delta t, x)\cdot\nabla R^i - \\
&\Big\{R^i\Big[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1((i-1)\Delta t, x, T_1, T_8)\Big] + W^i\Big\} \;=\; 0.
\end{aligned} \tag{16.178}$$

with

$$\begin{aligned}
&W^i := \frac{1}{\Delta t}\left(R^i(x,T_1,T_8) - R^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})\right) = \\
&\frac{1}{\Delta t}\left([R^i(x,T_1,T_8) - R^{i-1}(x,T_1,T_8)] + [R^{i-1}(x,T_1,T_8) - R^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})]\right) = \\
&\frac{1}{\Delta t}\left(Z^i(x,T_1,T_8) + [R^{i-1}(x,T_1,T_8) - R^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})]\right) = \\
&\frac{Z^i}{\Delta t} + \frac{1}{\Delta t}\Big(- B_1^{i-1}(x,T_1,T_8)\cdot\gamma^{i-1}(x,T_1)\Delta t - B_8^{i-1}(x,T_1,T_8)\cdot\delta^{i-1}(x,T_8)\Delta t - \\
&\textstyle\sum_{k,l=1,8}(1-\theta)B_{kl}^{i-1}(x,\tau_1^{i-1}\theta(x,T_1),\tau_8^{i-1}\theta(x,T_1))(\tau_k^{i-1}-T_k)\cdot(\tau_l^{i-1}-T_l)\Big)
\end{aligned}$$

(see (16.108)). By (16.177), $\|\nabla Z^i\|/\Delta t < G_Z$, $\|\nabla H_1^i\|/\Delta t < G_{H1}$ and $\|\nabla H_8^i\|/\Delta t < G_{H8}$ (uniformly with respect to i), thus combining it with the results of sections 16.8 and 16.11, we conclude that the expression in the curly brackets has its $C^{1,0}$ norm of the order of $O(1)$. It follows from Lemma 15.5, by taking $l = 3$ and the integration power p sufficiently large that

$$\|R^i\|_{W_p^3} \le C_{3p}$$

where the constants C_{3p} are uniformly bounded for all p. By using the Sobolev imbedding theorem, we conclude that for all $\beta \in (0,1)$ there exists a constant $C_{2\beta}$ independent of $i \in \{1,\dots,n\}$ such that

$$\|R^i\|_{C^{2+\beta}(\Omega)} \le C_{R2\beta}. \tag{16.179}$$

Having these relations we can use the refined version of the Gagliardo-Nirenberg inequality (see [4]) to obtain higher order estimate for functions Z^i. Let us recall the result proved in [4].

Lemma 16.21. *Assume that the real numbers $s_1, s_2, s \ge 0$, $\theta \in (0,1)$ and $1 \le p_1, p_2, p \le \infty$ satisfy the relations $s = \theta s_1 + (1-\theta)s_2$, $\dfrac{1}{p} = \dfrac{\theta}{p_1} + \dfrac{1-\theta}{p_2}$. Suppose that the following condition* **does not hold***:*

$$\mathcal{P}:\ s_2 \ge 1 \text{ is an integer, } p_2 = 1 \text{ and } s_2 - s_1 \le 1 - \frac{1}{p_1}.$$

Then, for every $\theta \in (0,1)$, there exists a constant C, depending on $s_1, s_2, p_1, p_2, \theta$ and Ω such that

$$\|f\|_{W^{s,p}(\Omega)} \le C\|f\|^{\theta}_{W^{s_1,p_1}(\Omega)}\|f\|^{1-\theta}_{W^{s_2,p_2}(\Omega)}, \quad \forall f \in W^{s_1,p_1}(\Omega)\cap W^{s_2,p_2}(\Omega).$$

Remark The refinement of the above result with respect to the classical formulation of the Gagliardo-Nirenberg inequality consists in the fact that the numbers s_1, s_2 and s may not be integers. □

Let us apply Lemma 16.21 for $f = Z'^i$. Thus, using (16.177) and Eq.(16.178), we conclude that there exists a constant $C_{2'}$ such that, for any $p_3 \in [1, \infty)$,

$$\|Z'^i\|_{W^{2,p_3}} < C_{2'}.$$

(As before, for simplicity, for fixed $k \in \{1, 2, 3\}$, $' := \dfrac{\partial}{\partial x_k}$.)

Thus, we take $s_2 = 2$, $s_1 = 0$, $p_1 = \infty$ and $p_2 = p_3$. Then $s_2 - s_1 \not\leq 1 - \dfrac{1}{p_1} = 1$, hence condition $\mathcal{P}$ does not hold. Taking $\theta \in (0, 1/2)$, we have $s_2 = 2(1-\theta) > 1$, $p > p_3$ and

$$\|f\|_{W^{2(1-\theta),p}(\Omega)} \leq C\|f\|^{\theta}_{L^\infty(\Omega)}\|f\|^{1-\theta}_{W^{2,p_3}(\Omega)}.$$

As p_3 is at our disposal, then we can find the smallest p_3 such that that $(2(1-\theta) - \dim(\Omega)/p)/2 = 1/2 - \theta$. For example taking $\theta = 2/7$ and $p_3 > 7dim(\Omega)$, we obtain $2(1-\theta) - dim(\Omega)/p > 9/7$, hence we conclude, by using the Sobolev's embedding theorem, that

$$\|Z'^i\|_{C^{9/7}} < C(\Delta t)^{2/7}, \tag{16.180}$$

which results in the estimate

$$\|Z^i\|_{C^{2+2/7}} < C(\Delta t)^{2/7}. \tag{16.181}$$

Similar reasoning can be applied to the functions H'^i_1 and H'^i_8. For $m = 1, 8$, we can thus obtain the inequalities of the form:

$$\|H'^i_m\|_{C^{9/7}} < C(\Delta t)^{2/7},$$

and consequently

$$\|H^i_m\|_{C^{2+2/7}} < C(\Delta t)^{2/7}. \tag{16.182}$$

16.20 Estimate of differences $Z^i - Z^{i-1}$

Note that Eq.(16.102) can be written as

$$\begin{aligned}
&d_R\nabla^2\left(Z^i(x,T_1,T_8) - Z^{i-1}(x,T_1,T_8)\right) - \frac{Z^i(x;T_1,T_8) - Z^{i-1}(x,T_1,T_8)}{\Delta t} - \\
&\frac{Z^{i-1}_*(x,T_1,T_8)}{\Delta t} - \left\{Z^i\left[\frac{\partial}{\partial T_1}\left(\gamma(c^{u;i-1}_{1*}, c^{u;i-1}_{8*}, T_1)\right) + \frac{\partial}{\partial T_8}\left(\delta(c^{u;i-1}_{8*}, T_8)\right) + F_1((i-1)\Delta t, x, T_1, T_8)\right]\right\} \\
&+R^{i-1}\Delta V^i + d_R\nabla^2 Z^{i-1}(x,T_1,T_8) + F_0((i-1)\Delta t, x)\cdot\nabla Z^{i-1} \;=\; 0
\end{aligned} \tag{16.183}$$

Now, using (16.177) and (16.181), we can use the maximum principle to conclude that

$$\frac{\|Z^i - Z^{i-1}\|_{C^0(\Omega)}}{\Delta t} = \left\|\frac{Z^i}{\Delta t} - \frac{Z^{i-1}}{\Delta t}\right\|_{C^0(\Omega)} \leq C_{diff}(\Delta t)^{2/7}. \tag{16.184}$$

Similarly, differentiating Eq.(16.183) with respect to T_m, $m = 1, 8$, (or rewriting Eq.(16.114) and using (16.177) and (16.182) we obtain by applying the maximum principle the inequalities:

$$\frac{\|H^i_m - H^{i-1}_m\|_{C^0(\Omega)}}{\Delta t} = \left\|\frac{H^i_m}{\Delta t} - \frac{H^{i-1}_m}{\Delta t}\right\|_{C^0(\Omega)} \leq C_{diff}(\Delta t)^{2/7}. \tag{16.185}$$

17 Convergence of the sequences as $\Delta t \to 0$

Let us write Eq.(16.5) in the form:

$$d_R \nabla^2 R^i + F_0(x)\cdot \nabla R^i - \frac{Z^i}{\Delta t} -$$
$$\Big\{ R^i \left[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1}, c_{8*}^{u;i-1}, T_1)\Big) + \frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1}, T_8)\Big) + F_1((i-1)\Delta t, x, T_1, T_8)\right] - \widetilde{W}^i \Big\} \; = \; 0, \tag{17.1}$$

where, according to Lemma 16.15,

$$\widetilde{W}^i := \frac{1}{\Delta t}\,[R^{i-1}(x,T_1,T_8) - R^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})] =$$
$$\frac{1}{\Delta t}\Big(- B^{i-1}_{\tau_1}(x,T_1,T_8)\cdot \gamma^{i-1}(x,T_1)\Delta t - B^{i-1}_{\tau_8}(x,T_1,T_8)\cdot \delta^{i-1}(x,T_8)\Delta t -$$
$$\textstyle\sum_{k,l=1,8}(\tau_k^{i-1} - T_k)\cdot(\tau_l^{i-1}-T_l)\int_0^1 B^{i-1}_{\tau_k\tau_l}\left(x, T_1 + s(\tau_1^{i-1}(x,T_1)-T_1), T_8 + s(\tau_8^{i-1}(x,T_8)-T_8)\right)(1-s)ds\Big)$$
$$= -B_1^{i-1}(x,T_1,T_8)\cdot\gamma^{i-1}(x,T_1)\Delta t - B_8^{i-1}(x,T_1,T_8)\cdot\delta^{i-1}(x,T_8) + O(\Delta t).$$

For given $n \ge 1$, $\Delta t = Tn^{-1} > 0$ and the set of points $\{t_{i_t} := i_t\Delta t\}_{i_t=1}^{i_t=n}$, $i_t \in \{0,\ldots,n\}$, let us define the function

$$\mathcal{R}^n(t,x,T_1,T_8) := R^{i_t-1}(x,T_1,T_8) +$$
$$(\Delta t)^{-1}(t - t_{i_t-1})\Big(R^{i_t}(x,T_1,T_8) - R^{i_t-1}(x,T_1,T_8)\Big) \quad \text{for } t_{i_t-1} \le t < t_{i_t}. \tag{17.2}$$

Below, we will analyse in sequence, the result of action of different operators of the left hand side of Eq.(16.5) on the function $\mathcal{R}^n$. For $t_{i_t-1} \le t \le t_{i_t}$, we thus have:

$$\Delta\mathcal{R}^n(t,x,T_1,T_8) = \Delta R^{i_t-1}(x,T_1,T_8) + (t - t_{i_t-1})\Big(\Delta R^{i_t}(x,T_1,T_8) - \Delta R^{i_t-1}(x,T_1,T_8)\Big)\cdot(\Delta t)^{-1}, \tag{17.3}$$

and, according to (16.181),

$$\|\Delta R^{i_t}(\cdot,T_1,T_8) - \Delta R^{i_t-1}(\cdot,T_1,T_8)\|_{C_x^{2/7}} \le C_\Delta(\Delta t)^{2/7}, \tag{17.4}$$

uniformly in (T_1,T_8). Then, by definition (17.2), for $t \in [t_{i-1}, t_i]$,

$$\frac{\partial \mathcal{R}^n}{\partial t} = (\Delta t)^{-1}\Big(R^{i_t}(x,T_1,T_8) - R^{i_t-1}(x,T_1,T_8)\Big).$$

This function is constant on each interval $t \in [t_{i-1}, t_i]$ and, according to (16.181), the difference between these values is of the order of $O(\Delta t)$. Next, for $t \in [t_{i-1}, t_i]$

$$
\begin{aligned}
&\frac{\partial}{\partial T_1}\left(\gamma(c_1^u(t,x),c_8^u(t,x),T_1)\,\mathcal{R}^n\right)+\frac{\partial}{\partial T_8}\left(\delta(c_8^u(t,x),T_8)\,\mathcal{R}^n\right)=\\
&\mathcal{R}^n\cdot\frac{\partial}{\partial T_1}\gamma(c_1^u(t,x),c_8^u(t,x),T_1)+\mathcal{R}^n\cdot\frac{\partial}{\partial T_8}\delta(c_8^u(t,x),T_8)+\\
&+\gamma(c_1^u(t,x),c_8^u(t,x),T_1)\cdot\frac{\partial}{\partial T_1}\mathcal{R}^n(t,x,T_1,T_8)+\delta(c_8^u(t,x),T_8)\cdot\frac{\partial}{\partial T_8}\mathcal{R}^n(t,x,T_1,T_8)=\\
&\Big[\mathcal{R}^n\cdot\frac{\partial}{\partial T_1}\Big(\gamma(c_1^u(t,x),c_8^u(t,x),T_1)-\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)\Big)\\
&+\mathcal{R}^n\cdot\frac{\partial}{\partial T_8}\Big(\delta(c_8^u(t,x),T_8)-\delta(c_{8*}^{u;i-1}(x),T_8)\Big)+\\
&\frac{\partial}{\partial T_1}\mathcal{R}^n(t,x,T_1,T_8)\Big(\gamma(c_1^u(t,x),c_8^u(t,x),T_1)-\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)\Big)+\\
&\frac{\partial}{\partial T_8}\mathcal{R}^n(t,x,T_1,T_8)\Big(\delta(c_8^u(t,x),T_8)-\delta(c_{8*}^{u;i-1}(x),T_8)\Big)\Big]+\\
&\Big\{\mathcal{R}^n\cdot\frac{\partial}{\partial T_1}\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)+\mathcal{R}^n\cdot\frac{\partial}{\partial T_8}\delta(c_{8*}^{u;i-1}(x),T_8)\Big\}_1+\\
&\Big\{\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)\cdot\frac{\partial}{\partial T_1}\mathcal{R}^n(t,x,T_1,T_8)+\delta(c_{8*}^{u;i-1}(x),T_8)\cdot\frac{\partial}{\partial T_8}\mathcal{R}^n(t,x,T_1,T_8)\Big\}_2
\end{aligned}
\tag{17.5}
$$

where $c_1^u(t,x)$ and $c_8^u(t,x)$ are defined by (16.45). Due to Lemma 16.19 (inequalities (16.151)), (16.152), together with the boundedness of the function $\mathcal{R}^n$ (implied by the boundedness of R^m, $m\in\{1,\dots,n\}$) and the boundedness of the functions B_1^m, B_8^m, it is seen that the expression in the square bracket $[\cdot]$ above is of the order of $\Delta t\, O(1)$ as $\Delta t\to 0$.

By (17.2), the first term in the curly bracket $\{\cdot\}_1$ equals:

$$
\Big(R^{i_t-1}(x,T_1,T_8)+(\Delta t)^{-1}(t-t_{i_t-1})\left(R^{i_t}(x,T_1,T_8)-R^{i_t-1}(x,T_1,T_8)\right)\Big)\cdot\frac{\partial}{\partial T_1}\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)
$$

$$
=R^{i_t-1}(x,T_1,T_8)\cdot\frac{\partial}{\partial T_1}\gamma(c_{1*}^{u;i-1}(x),c_{8*}^{u;i-1}(x),T_1)+O(1)\Delta t,
$$

where the last estimate is obtained via the boundedness of of the derivative of γ (see Lemma 16.7) and inequalities (16.144). Likewise,

$$
\mathcal{R}^n\cdot\frac{\partial}{\partial T_8}\delta(c_{8*}^{u;i-1}(x),T_8)=R^{i_t-1}(x,T_1,T_8)\cdot\frac{\partial}{\partial T_1}\delta(c_{8*}^{u;i-1}(x),T_1)+O(1)\Delta t,
$$

Similarly, the expression in the bracket $\{\cdot\}_2$ equals

$$
\begin{aligned}
&\frac{1}{\Delta t}\cdot\Big(\Delta t\cdot\gamma(c_1^u(t,x),c_8^u(t,x),T_1)\cdot\frac{\partial}{\partial T_1}\mathcal{R}^n(t,x,T_1,T_8)+\Delta t\cdot\delta(c_8^u(t,x),T_8)\cdot\frac{\partial}{\partial T_8}\mathcal{R}^n(t,x,T_1,T_8)\Big)=\\
&\frac{1}{\Delta t}\cdot\Big(\Delta t\cdot\Big(\gamma(c_1^u(t,x),c_8^u(t,x),T_1)-\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big)\cdot\frac{\partial}{\partial T_1}\mathcal{R}^n(t,x,T_1,T_8)+\\
&\Delta t\cdot\Big(\delta(c_8^u(t,x),T_8)-\delta(c_{8*}^{u;i-1},T_8)\Big)\cdot\frac{\partial}{\partial T_8}\mathcal{R}^n(t,x,T_1,T_8)\Big)+\\
&\frac{1}{\Delta t}\cdot\Big(\Delta t\cdot\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\cdot\frac{\partial}{\partial T_1}\mathcal{R}^n(t,x,T_1,T_8)+\Delta t\cdot\delta(c_{8*}^{u;i-1},T_8)\cdot\frac{\partial}{\partial T_8}\mathcal{R}^n(t,x,T_1,T_8)\Big)=\\
&-\frac{1}{\Delta t}\cdot\Big(\mathcal{R}^n(t,x,T_1-\Delta t\cdot\gamma,T_8-\Delta t\cdot\delta)-\mathcal{R}^n(t,x,T_1,T_8)+(\Delta t)^2\,O(1)\Big),
\end{aligned}
$$

what follows from inequality (16.151) together with the boundedness of the derivatives of the function $\mathcal{R}^n$ with respect to T_1 and T_8 (implied by the boundedness of the functions B_1^m, B_8^m).

Now, using the definition (17.2), for $t\in[t_{i-1},t_i]$, denotations (16.105) and (16.106)), and the results of section 16.15 (see (16.144)), we have:

$$
\begin{aligned}
&\frac{1}{\Delta t}\Big(\mathcal{R}^n(t,x,T_1-\Delta t\cdot\gamma,T_8-\Delta t\cdot\delta)-\mathcal{R}^n(t,x,T_1,T_8)\Big)=\\
&\frac{1}{\Delta t}\Big\{R^{i-1}(t,x,T_1-\Delta t\cdot\gamma^{i-1},T_8-\Delta t\cdot\delta^{i-1})-R^{i-1}(t,x,T_1,T_8)+\\
&(\Delta t)^{-1}(t-t_{i_t-1})\Big(\Big[R^i(t,x,T_1-\Delta t\cdot\gamma^i,T_8-\Delta t\cdot\delta^i)-R^i(t,x,T_1,T_8)\Big]-\\
&\qquad\Big[R^{i-1}(t,x,T_1-\Delta t\cdot\gamma^{i-1},T_8-\Delta t\cdot\delta^{i-1})-R^{i-1}(t,x,T_1,T_8)\Big]\Big)\Big\}
\end{aligned}
\tag{17.6}
$$

Now, according to (16.109), the absolute value of the expression

$$
\Big[R^i(t,x,T_1-\Delta t\cdot\gamma^i,T_8-\Delta t\cdot\delta^i)-R^i(t,x,T_1,T_8)\Big]-\Big[R^{i-1}(t,x,T_1-\Delta t\cdot\gamma^{i-1},T_8-\Delta t\cdot\delta^{i-1})-R^{i-1}(t,x,T_1,T_8)\Big]
$$

is bounded from above by $\Big(|H_1^{i-1}|\overline{\gamma}+|H_8^{i-1}|\overline{\delta}\Big)\Delta t+r_1(\Delta t)^2$ for some constant r_1 independent of i, hence by Lemma 16.18 is of the order of $O((\Delta t)^2)$. It follows that

$$
\begin{aligned}
&\frac{1}{\Delta t}\Big(\mathcal{R}^n(t,x,T_1-\Delta t\cdot\gamma,T_8-\Delta t\cdot\delta)-\mathcal{R}^n(t,x,T_1,T_8)\Big)=\\
&\frac{1}{\Delta t}\Big\{R^{i-1}(t,x,T_1-\Delta t\cdot\gamma^{i-1},T_8-\Delta t\cdot\delta^{i-1})-R^{i-1}(t,x,T_1,T_8)\Big\}+O(\Delta t).
\end{aligned}
$$

Finally, for $t\in[t_{i-1},t_i]$, we have:

$$
\begin{aligned}
&F_0(x)\cdot\nabla\mathcal{R}^n(t,x,T_1,T_8)=\\
&F_0(x)\cdot\nabla R^{i_t}(x,T_1,T_8)+[(t-t_{i_t-1})\cdot(\Delta t)^{-1}-1]\,F_0(x)\cdot\Big(\nabla R^{i_t}(x,T_1,T_8)-\nabla R^{i_t-1}(x,T_1,T_8)\Big)=\\
&F_0(x)\cdot\nabla R^{i_t}(x,T_1,T_8)+\Big[(t-t_{i_t-1})\cdot(\Delta t)^{-1}-1\Big]\,F_0(x)\cdot\Big(\nabla Z^i(x,T_1,T_8)\Big).
\end{aligned}
\tag{17.7}
$$

By (16.177) the last term vanishes as fast as Δt for $\Delta t\to 0$. It thus follows that for $t\in[t_{i-1},t_i]$ we can write

$$
\begin{aligned}
&d_R\nabla^2\mathcal{R}^n-\frac{\partial\mathcal{R}^n}{\partial t}-\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\,\mathcal{R}^n\Big)-\frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1},T_8)\,\mathcal{R}^n\Big)+\\
&F_0(x)\cdot\nabla\mathcal{R}^n-F_1((i-1)\Delta t,x,T_1,T_8)\,\mathcal{R}^n=\\
&d_R\nabla^2R^i-\Big\{R^i\Big[\frac{\partial}{\partial T_1}\Big(\gamma(c_{1*}^{u;i-1},c_{8*}^{u;i-1},T_1)\Big)+\frac{\partial}{\partial T_8}\Big(\delta(c_{8*}^{u;i-1},T_8)\Big)+F_1((i-1)\Delta t,x,T_1,T_8)+\frac{1}{\Delta t}\Big]\Big\}+\\
&\frac{R^{i-1}(x;\tau_1^{i-1},\tau_8^{i-1})}{\Delta t}+F_0(x)\cdot\nabla R^i+(\Delta t)^{2/7}O(1)=0+(\Delta t)^{2/7}O(1).
\end{aligned}
$$

Let us consider the convergence properties of the sequence $\mathcal{R}^n$. We will use the Arzela-Ascoli lemma in the spaces $W_q^{2,1}(M_T)$, $M_T=\Omega\times(0,T)\times\mathcal{A}_{TT}$, where $\mathcal{A}_{TT}$ is any open simply connected set with a smooth boundary comprising the support of the functions R^i for all $i\in\{1,\ldots,n\}$ with the norm defined by

$$
\|u\|_{q,Q_T}^{(2)}=\sum_{j=0}^{2}<<u>>_{q,Q_T}^{(j)}
$$

where (cf. (1.3),(1.4) in I.1 of [23])

$$<< u >>^{(0)}_{q,Q_T} = \|u\|_{q,Q_T}, \quad << u >>^{(1)}_{q,Q_T} = \sum_{k=1,2,3} \Big\| \frac{\partial u}{\partial x_k} \Big\|_{q,Q_T} + \sum_{l=1,8} \Big\| \frac{\partial u}{\partial T_l} \Big\|_{q,Q_T},$$

$$<< u >>^{(2)}_{q,Q_T} = \sum_{k,s=1,2,3} \Big\| \frac{\partial^2 u}{\partial x_k \partial x_s} \Big\|_{q,Q_T} + \sum_{l,m=1,8} \Big\| \frac{\partial^2 u}{\partial T_l \partial T_m} \Big\|_{q,Q_T} + \Big\| \frac{\partial u}{\partial t} \Big\|_{q,Q_T},$$

and

$$\|f\|_{q,Q_T} = \Big(\int_0^T \Big(\int_{\Omega \times \mathcal{A}_{TT}} |f(t,x,T_1,T_8)|^q \, dx dT_1 dT_8 \Big) dt \Big)^{1/q}.$$

According to the estimates derived in the previous sections, for each $n \in \mathbb{N}$, the norm of the functions $\mathcal{R}^n$ have their $W^{2,1}_{q,Q_T}$ uniformly bounded for any arbitrarily large $q > 0$. As it follows from the Corollary after Theorem 9.1 of section IV.9 (which is based on Lemma 3.3 of chapter II) in [23], for all $n \in \mathbb{N}$, the functions $\mathcal{R}^n$ satisfy the norm inequality

$$\|\mathcal{R}^n\|^{(2-\Upsilon)} \leq C_\Omega \|\mathcal{R}^n\|^{(2)}_{q,Q_T}, \quad \Upsilon = \frac{dim(\Omega \times A_{TT}) + 2}{q},$$

for some constant C_Ω, where $\|\cdot\|^{\chi}$ denotes the Hölder norm $\|\cdot\|^{\chi/2,\chi}_{t,x}$. It follows that the functions $\mathcal{R}^n$ are uniformly bounded in the $C^{\tilde{\mu}(q)/2,\tilde{\mu}(q)}_{t,(x,T_1,T_8)}$ norm, with $\tilde{\mu}(q)$ satisfying the inequality

$$\tilde{\mu}(q) < 2 - \frac{dim(\Omega \times \mathcal{A}_{TT})}{q} = 2 - \frac{7}{q},$$

hence $\tilde{\mu}(q) > 1 + \beta$, for any $\beta \in (0,1)$ if q is sufficiently large. Now, by the Arzela-Ascoli lemma, from the sequence $\{\mathcal{R}^n\}_{n=1}^{\infty}$, we can choose a subsequence converging to a function $\mathcal{R} \in C^{\mu/2,\mu,1+\mu}_{t,x,(T_1,T_8)}([0,T] \times \overline{\Omega} \times \overline{\mathbb{R}^2_+})$ for any $\mu < \tilde{\mu}(q)$ (cf. section 16.12). Simultaneously, as $n \to \infty$, then the functions c_1^u and c_8^u tend along the appropriate subsequence (being, in general, a subsequence of the subsequence along which $\{\mathcal{R}^n\}_{n=1}^{\infty}$ converges), to some functions c_{1D}^u and c_{8D}^u belonging to the space $C^{(1+\beta)/2,1+\beta}_{t,x}([0,T] \times \overline{\Omega})$ for any $\beta \in (0,1)$. Now, for fixed (T_1, T_8) and for every $n < \infty$, we can multiply the equation satisfied by $\mathcal{R}^n$ by a smooth test function $\phi^*(t,x)$, integrate by parts and consider it as an equation for weak solutions $\mathcal{R}^n$ as a function of (t,x). By passing to the limit $\Delta t \to 0$, we conclude that $\mathcal{R}$ is a weak solution to the equation:

$$\frac{\partial \mathcal{R}}{\partial t} = d_R \nabla^2 \mathcal{R} + f(t,x,T_1,T_8) \tag{17.8}$$

where

$$f(t,x,(T_1,T_8)) = F_0(x) \cdot \nabla \mathcal{R} +$$

$$\Big(- \frac{\partial}{\partial T_1} \left(\gamma(c_{1D}^u(t,x), c_{8D}^u(t,x), T_1) \, \mathcal{R} \right) - \frac{\partial}{\partial T_8} \left(\delta(c_{8D}^u(t,x), T_8) \, \mathcal{R} \right) - F_1(t,x,T_1,T_8) \, \mathcal{R} \Big) \in C^{\mu/2,\mu}_{t,x}$$

with (T_1, T_8) treated as parameters. Let us note that given the function $f(t,x)$, Eq.(17.8), supplemented with the homogeneous Neumann boundary conditions and $C^{4,4}_{x,(T_1,T_8)}$ initial conditions (as it was supposed in Assumption 16.4) has a solution with finite $C^{1+\mu/2,2+\mu,2+\mu}_{t,x,(T_1,T_8)}$ norm. This solution is unique. For, suppose that it is not true, and that, for fixed (T_1, T_8), there exists another solution $\mathcal{P}$ to this equation (with the same boundary and initial conditions). By subtracting, multiplying by $\mathcal{D} = \mathcal{R} - \mathcal{P}$ and integrating by parts it is seen that $\mathcal{D}$ satisfies the equation

$$\frac{1}{2} \frac{\partial}{\partial t} \int_\Omega \mathcal{D}^2(t,x,T_1,T_8) dx = -d_R \int_\Omega |\nabla \mathcal{D}(t,x,T_1,T_8)|^2 dx.$$

As $\mathcal{D}(0,x,T_1,T_8) \equiv 0$, we conclude that $\mathcal{D}(t,x,T_1,T_8) \equiv 0$.

We are thus in a position to formulate the summarizing theorem of Part III.

Theorem 17.1. *The triple $(\mathcal{R}, c^u_{1D}, c^u_{8D})$ satisfies system (16.2)-(16.4). The function*

$$\mathcal{R} \in C^{1+\mu/2,2+\mu}_{t,(x,T_1,T_8)}((0,T)\times\overline{\Omega}\times\overline{\mathbb{R}^2_+}),$$

whereas the functions $c^u_{1D},\ c^u_{8D} \in C^{1+\mu/2,2+\mu}_{t,x}((0,T)\times\Omega)\cap C(([0,T]\times\overline{\Omega})$.

Proof The fact that $\mathcal{R}, c^u_{1D}, c^u_{8D} \in C^{1+\mu/2,2+\mu}_{t,(x,T_1,T_8)}((0,T)\times\overline{\Omega}\times\overline{\mathbb{R}^2_+})$ has been shown above. We also showed that these functions satisfy Eq. (16.5). We will prove that they satisfy Eqs (16.6)-(16.7). Let us consider the second equation in system (16.5)-(16.7). For $t\in[(i-1)\Delta t, i\Delta t)$ it can be written in the form

$$\frac{\partial c^{u;i}_1}{\partial t} = \nabla^2 c^{u;i}_1 + \widetilde{\nu}\int_0^\infty\int_0^\infty c^{8;i-1}_{8*} T_8\,\mathcal{R}^n\,dT_1\,dT_8 - c^{u;i}_1 - \widetilde{\nu}\int_0^\infty\int_0^\infty c^{8;i-1}_{8*} T_8\,\delta_{Rni}\,dT_1\,dT_8 \tag{17.9}$$

where $\delta_{Rni} = (\Delta t)^{-1}(t-t_{i_t-1})\Big(R^{i_t}(x,T_1,T_8) - R^{i_t-1}(x,T_1,T_8)\Big)$. As, independently of t and $\Delta t = Tn^{-1}$, $(\Delta t)^{-1}(t-t_{i_t-1})\le 1$, then using Lemmata 16.18 and 16.20, we conclude that for each (T_1,T_8),

$$\lim_{n\to\infty}\|\delta_{Rni}(\cdot,\cdot,(T_1,T_8))\|_{C^{0,\mu}_{t,x}} = 0.$$

By using the functions defined by (16.45), for fixed n, the set of equations (17.9) can be written as the equation

$$\frac{\partial c^u_1}{\partial t} = \nabla^2 c^u_1 + \widetilde{\nu}\int_0^\infty\int_0^\infty c^8_{8*}(c^8_u) T_8\,\mathcal{R}^n\,dT_1\,dT_8 - c^{u;i}_1 - \widetilde{\nu}\int_0^\infty\int_0^\infty c^8_{8*}(c^8_u) T_8\,\delta_{Rni}\,dT_1\,dT_8. \tag{17.10}$$

In fact, we should consider c^u_1 as a weak solution to Eq.(17.10). Let us note that $\int_0^\infty\int_0^\infty c^8_{8*}(c^8_u)T_8\,\mathcal{R}^n\,dT_1\,dT_8$ is a continuous function of (t,x), whereas the function $\int_0^\infty\int_0^\infty c^8_{8*}(c^8_u)T_8\,\delta_{Rni}\,dT_1\,dT_8$ is of L^∞ class. It follows from Theorem 16.11 that the functions c^u_1 and c^u_8 have finite $C^{(1+\beta)/2,1+\beta}_{t,x}$ norm for all $\beta\in(0,1)$. Thus by considering a subsequence of $\{n\}_1^\infty$, for which all the considered sequences of functions converge, we conclude that c^u_{1D}, c^u_{8D} are of the class $C^{(1+\tilde\beta)/2,1+\tilde\beta}_{t,x}$, with $\tilde\beta\in(0,1)$ and, in fact, are weak solutions to the equation on $(0,T)\times\Omega$

$$\frac{\partial c^u_{1D}}{\partial t} = \nabla^2 c^u_{1D} - c^u_{1D} + f(t,x,T_1,T_8), \tag{17.11}$$

where

$$f(t,x) = \widetilde{\nu}\int_0^\infty\int_0^\infty c^8_{8D} T_8\,\mathcal{R}\,dT_1\,dT_8$$

and $c^8_{8D}(t,x) = c^u_{8D}(t,x)\,(1+c^u_{8D}(t,x))^{-1}$. As $f(t,x)\in C^{(1+\beta)/2,1+\beta}_{t,x}$ then using, e.g. [23, Theorem 5.3, chapter IV], we conclude that there exists a solution to Eq. 17.11, with the homogeneous Neumann boundary conditions and initial conditions in $C^4(\overline{\Omega})$ class as supposed in Assumption 16.4, has a solution in $C^{1+\beta/2,2+\beta}_{t,x}([0,T]\times\overline{\Omega})$ class. As $f(t,x)$ is fixed, then the solution is unique, what can be shown as in the case of the equation for $\mathcal{R}$. The third equation in system (16.5)-(16.7) can be considered in the similar way. □

18 Conclusions

In the dissertation, we used two different approaches to study the initial boundary value problems connected with system (1.11)-(1.13). In Part II, we considered scalar linear equations with the form of their differential part similar to that of Eq. (1.11), and despite its mixed parabolic-hyperbolic structure, we managed to construct explicit solutions in the homogeneous and inhomogeneous cases. Besides to the construction of solutions, an important result of this part presented in Lemma 9.5 (see also Lemma 11.1), states that, in a sense, these solutions can be treated as a limit of solutions with added diffusional terms with respect to the auxiliary variables T_1 and T_8. This result seems to be particularly significant in designing the method of numerical analysis of the model. In Part III we applied a modification of the Rothe method and proved the existence of solution to a simplified version of system (1.11)-(1.13) in the limit of the size of the step interval $\Delta t \to 0$. To obtain a priori estimates, necessary for the proof of convergence of the family of solutions, we use the maximum principle for elliptic equations. In derivation of these estimates we extensively took advantage of the celebrated paper [1]. It seems that this method can be used to general classes of similar systems. Its applicability depends on appropriate behaviour of characteristics to the hyperbolic part of the equation of the mixed type. In particular, we assumed that the projections of the characteristics onto the (T_1, T_8)-plane enter its positive quarter in the course of time.

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A Appendix A – Laplace operator in $\mathbb{R}^m$ in local coordinates connected with an $(n-1)$ dimensional hypersurface

Consider a hypersurface $S = \{(x_1, x_2, \ldots, x_n) : x_1 - \omega(x_2, \ldots, x_n) = 0\}$ defined in the vicinity of the point $\mathbf{x}_0 \in S$. Assume that the hyperplane $x_1 = 0$ is tangent to the hypersuface S at the point $\mathbf{x}_0$. It follows that

$$\frac{\partial \omega}{\partial x_i}(\mathbf{x}_0) = 0, \quad i = 2, \ldots, n. \tag{A.1}$$

Let

$$\xi = x_1 - \omega(x_2, \ldots, x_n) \quad \eta_i = x_i \quad \text{for } i = 2, \ldots, n. \tag{A.2}$$

Let us derive the expression for the Laplace operator Δ in the variables $(\xi, \eta_1, \ldots, \eta_n)$ at the point $\mathbf{x} = \mathbf{x_0}$.

Lemma A.1. *Suppose that (A.1) and (A.2) hold. Then at $x = x_0$:*

$$\Delta_{\xi,\eta_1,\ldots,\eta_n} = \Big(\frac{\partial^2}{\partial \xi^2} + \sum_{i=2,\ldots,n}^{n} \frac{\partial^2}{\partial \eta_i^2}\Big) + \sum_{i=2,\ldots,n}^{n} \kappa_k(\mathbf{x}_0)\frac{\partial}{\partial \xi}$$

where $\kappa_k(\mathbf{x}_0) := -\dfrac{\partial^2 \omega}{\partial x_k^2}(\mathbf{x}_0), \quad k = 2, \ldots, n$ *are the principal curvatures of the surface S at $\mathbf{x} = \mathbf{x}_0$.*

We have:

$$\frac{\partial}{\partial x_1} = \frac{\partial \xi}{\partial x_1}\frac{\partial}{\partial \xi} + \sum_{i=2}^{n} \frac{\partial \eta_i}{\partial x_1}\frac{\partial}{\partial \eta_i}.$$

and

$$\frac{\partial}{\partial x_k} = \frac{\partial \xi}{\partial x_k}\frac{\partial}{\partial \xi} + \sum_{i=2}^{n} \frac{\partial \eta_i}{\partial x_k}\frac{\partial}{\partial \eta_i}.$$

It follows that

$$\frac{\partial^2}{\partial x_1^2} = \frac{\partial \xi}{\partial x_1}\frac{\partial}{\partial \xi}\Big(\frac{\partial \xi}{\partial x_1}\frac{\partial}{\partial \xi} + \sum_{i=2}^{n} \frac{\partial \eta_i}{\partial x_1}\frac{\partial}{\partial \eta_i}\Big) + \sum_{i=2}^{n} \frac{\partial \eta_i}{\partial x_1}\frac{\partial}{\partial \eta_i}\Big(\frac{\partial \xi}{\partial x_1}\frac{\partial}{\partial \xi} + \sum_{j=2}^{n} \frac{\partial \eta_j}{\partial x_1}\frac{\partial}{\partial \eta_j}\Big)$$

and

$$\frac{\partial^2}{\partial x_k^2} = \frac{\partial \xi}{\partial x_k}\frac{\partial}{\partial \xi}\Big(\frac{\partial \xi}{\partial x_k}\frac{\partial}{\partial \xi} + \sum_{i=2}^{n} \frac{\partial \eta_i}{\partial x_k}\frac{\partial}{\partial \eta_i}\Big) + \sum_{i=2}^{n} \frac{\partial \eta_i}{\partial x_k}\frac{\partial}{\partial \eta_i}\Big(\frac{\partial \xi}{\partial x_k}\frac{\partial}{\partial \xi} + \sum_{j=2}^{n} \frac{\partial \eta_j}{\partial x_k}\frac{\partial}{\partial \eta_j}\Big)$$

Due to (A.2) we have in some vicinity of $\mathbf{x}_0$:

$$\frac{\partial \xi}{\partial x_1} = 1, \quad \frac{\partial \eta_i}{\partial x_1} = 0, \quad \frac{\partial \eta_i}{\partial x_k} = \delta_{ik}, \quad \text{for } i, k \in \{2, \ldots, n\},$$

and exactly at $\mathbf{x} = \mathbf{x}_0$, for $k = 2, \ldots, n$,

$$\frac{\partial \xi}{\partial x_k} = -\frac{\partial \omega}{\partial x_k} = 0.$$

We thus have at $\mathbf{x}_0$,

$$\frac{\partial^2}{\partial x_1^2} = \frac{\partial^2}{\partial \xi^2}$$

and, for $k = 2, \ldots, n$,

$$\frac{\partial^2}{\partial x_k^2} = \frac{\partial}{\partial \eta_k}\Big(\frac{\partial \xi}{\partial x_k}\frac{\partial}{\partial \xi}\Big) + \frac{\partial^2}{\partial^2 \eta_k} = -\frac{\partial}{\partial x_k}\Big(\frac{\partial \omega}{\partial x_k}\Big)\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial^2 \eta_k} = -\Big(\frac{\partial^2 \omega}{\partial x_k^2}\Big)\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial^2 \eta_k}.$$

Thus finally at $\mathbf{x} = \mathbf{x}_0$:

$$\Delta_{\xi,\eta_1,\dots,\eta_n} \;=\; \Big(\frac{\partial^2}{\partial \xi^2} \;+ \sum_{i=2,\dots,n}^{n} \frac{\partial^2}{\partial \eta_i^2}\Big) \;+\; \sum_{i=2,\dots,n}^{n} \kappa_k(\mathbf{x}_0)\frac{\partial}{\partial \xi}$$

where

$$\kappa_k(\mathbf{x}_0) := -\frac{\partial^2 \omega}{\partial x_k^2}(\mathbf{x}_0), \quad k = 2, \dots, n,$$

can be interpreted as the principal curvatures of the surface S at $\mathbf{x} = \mathbf{x}_0$.

Appendix B